

# **Weather Risk Management: CAT bonds and Weather Derivatives**

## **D I S S E R T A T I O N**

zur Erlangung des akademischen Grades

doctor rerum politicarum

(Doktor der Wirtschaftswissenschaft)

eingereicht an der

Wirtschaftswissenschaftlichen Fakultät

Humboldt-Universität zu Berlin

von

**Frau M.Sc. Brenda López Cabrera**

13.03.1980 in Puebla, Mexiko

Präsident der Humboldt-Universität zu Berlin:

Prof. Dr. Christoph Markschies

Dekan der Wirtschaftswissenschaftlichen Fakultät:

Prof. Oliver Guenther, Ph.D.

Gutachter:

1. Prof. Dr. Wolfgang Haerdle

2. Prof. Dr. Vladimir Spokoiny

**eingereicht am:** 17 März 2010

**Tag des Kolloquiums:** 27 April 2010

## **Abstract**

CAT bonds and weather derivatives are end-products of a process known as securitization that transform non-tradable (natural catastrophes or weather related) risk factors into tradable financial assets. As a result the markets for such products are typically incomplete. Since appropriate measures of the risk associated to a particular price become necessary for pricing, one essentially needs to incorporate the market price of risk (MPR), which is an important parameter of the associated equivalent martingale measure. The majority of papers so far has priced non-tradable assets assuming zero MPR, but this assumption yields biased prices and has never been quantified earlier. This thesis deals with the differences between historical and risk neutral behaviors of the non-tradable underlyings and gives insights into the behaviour of the market price of weather risk and weather risk premium. The thesis starts by introducing the risk transferring instruments, the financial - statistical techniques and ends up by examining the real data applications with particular focus on the implied trigger intensity rates of a parametric CAT bond for earthquakes and the MPR of temperature derivatives.

## **Zusammenfassung**

CAT-Bonds und Wetterderivate sind die Endprodukte eines Verbriefungsprozesses, der nicht handelbare Risikofaktoren (Wetterschäden oder Naturkatastrophenschäden) in handelbare Finanzanlagen verwandelt. Als Ergebnis sind die Märkte für diese Produkte in der Regel unvollständig. Da geeignete Risikomaße in Bezug auf einen bestimmten Preis Voraussetzung sind zur Preisbestimmung, ist es notwendig den Marktpreis des Risikos (MPR), welcher ein wichtiger Parameter des zugehörigen äquivalenten Martingalmaß ist, zu berücksichtigen. Die Mehrheit der bisherigen Veröffentlichungen haben die Preise der nicht handelbaren Vermögenswerte mittels der Annahme geschätzt, dass der MPR gleich null ist. Diese Annahme verzerrt allerdings die Preise und wurde bisher noch nicht quantifiziert. Diese Doktorarbeit beschäftigt sich mit den Unterschieden zwischen dem historischen und dem risikoneutralen Verhalten der nicht handelbaren Basiswerte und gibt Einblicke in den Marktpreis für Wetterrisiko und die Wetterrisikoprämie. Diese Arbeit beginnt mit einer Darstellung der Instrumente zur Übertragung der Risiken, gefolgt von den finanziellen - statistischen Verfahren und endet mit einer Untersuchung reeller Daten, wobei der Schwerpunkt auf die implizierten Trigger-Intensitätsraten eines parametrischen CAT-Bond für Erdbeben und auf den MPR der Temperatur Derivate gelegt wird.

# Acknowledgement

I would like to thank to Professor Dr. Wolfgang Haerdle for supervising and supporting me through the whole time of my Ph.D. studies. He introduced me to the world of financial statistics and encouraged me to work on the analysis of weather risk management.

I am thankful to Professor Dr. Spokoiny for willing accepting to evaluate my thesis and sit in the examination committee.

I would like to thank all those people with whom I collaborated during the preparation of the thesis. The theoretical part of the thesis is based on the results of close cooperation with Professor Fred Espen Benth, whose extraordinary deep knowledge and experience in financial mathematics and energy markets helped me a lot in understanding of these new methods. I also appreciate him the discussions and comments to improve the estimation algorithms and hospitality during my visits at the University of Oslo.

I am grateful to Professor Jianqing Fan for inviting me to come to Princeton University and giving me valuable suggestions.

I owe much to many colleagues and researchers for sharing their time with me by numberless discussions and consultations during my work, among other these were: Szymon Borak, Enzo Giacomini, Jelena Bradic, and of course my thanks goes to all members of the Institute for Statistics at Humboldt University, C.A.S.E. and CRC 649 for friendly atmosphere and encouragement. I gratefully acknowledge the financial support from NaFOEG - Promotionsfoerderung and the Deutsche Forschungsgemeinschaft via CRC 649 Oekonomisches Risiko, Humboldt-Universitaet zu Berlin.

Last but certainly not least I am deeply indebted to my family for their constant support.

Berlin, March 16, 2010.

Brenda López Cabrera

# Contents

<b>Acknowledgement</b>	<b>iv</b>
<b>1 Introduction</b>	<b>1</b>
<b>2 Theoretical Background</b>	<b>6</b>
2.1 Stochastic Analysis . . . . .	6
2.2 Stochastic price modelling . . . . .	10
2.3 Pricing futures on the spot market . . . . .	13
<b>3 Catastrophe (CAT) Bonds</b>	<b>17</b>
3.1 Definitions . . . . .	17
3.2 Calibrating a Mexican Parametric CAT Bond . . . . .	19
3.2.1 Calibration in the Reinsurance Market . . . . .	22
3.2.2 Calibration in the Capital Market . . . . .	23
3.2.3 Calibration via Historical data . . . . .	24
3.3 Pricing modelled-index CAT bonds for Mexican earthquakes . . . . .	28
3.3.1 Severity of Mexican earthquakes . . . . .	29
3.3.2 Frequency of Mexican earthquakes . . . . .	34
3.3.3 Pricing modelled-Index CAT bonds . . . . .	35
3.4 Conclusion . . . . .	42
<b>4 Weather Derivatives</b>	<b>44</b>
4.1 Definitions . . . . .	45
4.2 Modelling Temperature . . . . .	47
4.2.1 Properties of temperature data . . . . .	47
4.2.2 An Ornstein-Uhlenbeck driven by a Fractional Brownian Motion . . . . .	48
4.2.3 An Ornstein-Uhlenbeck Model driven by a Brownian motion . . . . .	49
4.2.4 An Ornstein-Uhlenbeck Model driven by a Lévy Process . . . . .	49
4.2.5 Empirical Analysis of Temperature Dynamics . . . . .	50
4.2.6 Localizing temperature residuals . . . . .	61
4.3 Stochastic Pricing model . . . . .	68
4.4 The implied market price of weather risk . . . . .	74
4.4.1 Constant market price of risk for different daily contract . . . . .	74
4.4.2 Constant market price of risk per trading day . . . . .	75
4.4.3 Two constant market prices of risk per trading day . . . . .	75
4.4.4 General form of the market price of risk per trading day . . . . .	76
4.4.5 Bootstrapping the market price of risk . . . . .	77

## *Contents*

4.4.6	Smoothing the market price of risk over time . . . . .	79
4.4.7	Statistical and economical insights of the MPR . . . . .	80
4.4.8	Pricing CAT-HDD-CDD futures . . . . .	85
4.5	The risk premium and the market price of weather risk . . . . .	85
4.6	Temperature baskets . . . . .	90
4.6.1	Basket indices . . . . .	90
4.6.2	Stochastic modelling for Basket temperatures . . . . .	91
4.6.3	Pricing of Basket temperatures . . . . .	92
4.7	Conclusions and further research . . . . .	95
<b>Bibliography</b>		<b>97</b>

# List of Figures

3.1	Cash flows diagram of a CAT bond . . . . .	18
3.2	Number of Mexican earthquakes occurred during 1900-2003 . . . . .	20
3.3	Map of seismic regions in Mexico. . . . .	21
3.4	The cash flows diagram for the Mexican CAT bond . . . . .	22
3.5	Magnitude of trigger events . . . . .	26
3.6	Historical and modelled losses of Mexican earthquakes (in million dollars) occurred in Mexico during 1900-2003 and without outliers of the earthquakes in 1985 and 1999 . . . . .	31
3.7	The log of the empirical mean excess function $\log \left\{ \hat{e}_{n(x)} \right\}$ for the modelled loss data with and without the outlier of the earthquake in 1985. . .	32
3.8	The log of the empirical limited expected value function $\log \left\{ \hat{l}_{n(x)} \right\}$ and $\log (l_x)$ for the log-normal, Pareto, Burr, Weibull and Gamma distributions for the modelled loss with and without the outlier of the 1985 earthquake . . . . .	34
3.9	The log of the empirical mean excess function $\log \left\{ \hat{e}_{n(t)} \right\}$ for the earthquake data and the $\log (e_t)$ for the log-normal, exponential, Pareto and Gamma distributions for the earthquakes data . . . . .	35
3.10	The accumulated number of earthquakes (solid blue line) and mean value functions $E(N_t)$ of the Homogeneous Poisson Process (HPP) with the constant intensity $\lambda = 1.8504$ (solid black line) and the time dependent intensity $\lambda_s = 1.8167$ (dashed red line) . . . . .	37
3.11	Coupon CAT bond prices (vertical axis) with respect to the threshold level (horizontal right axis) and expiration time (horizontal left axis) under the Burr distribution and a Homogeneous Poisson Process . . . . .	39
3.12	The (Zero) Coupon CAT bond prices ((left) right panel) at time to maturity $T = 3$ years with respect to the threshold level $D$ . The CAT bond prices under the Burr distribution (solid lines), the Pareto distribution (dotted lines) and under different loss models (different color lines) . . .	42
4.1	Average daily temperatures, the Fourier truncated and the local linear seasonal component for different cities. . . . .	53
4.2	PACF of detrended temperatures for different cities. . . . .	54
4.3	Residuals of daily temperatures, Squared residuals for different cities. .	55
4.4	Residuals of daily temperatures, Squared residuals for different cities. .	56

## List of Figures

4.5	ACF of Residuals of daily temperatures $\varepsilon_t$ (left panels), Squared residuals $\varepsilon_t^2$ (right panels) for different cities. . . . .	57
4.6	Daily empirical variance (black line), the Fourier truncated (dashed line) and the local linear smoother seasonal variation (gray line) for different cities. . . . .	59
4.7	ACF of Residuals of daily temperatures $\varepsilon_t$ (left panels), Squared residuals $\varepsilon_t^2$ (right panels) after dividing out the seasonal volatility $\hat{\sigma}_{t,LLR}^2$ from the regression residuals for different cities. . . . .	60
4.8	Log of Normal Kernel (stars) and Log of Kernel smoothing density estimate of standardized residuals $\hat{\varepsilon}_t/\hat{\sigma}_{t,LLR}$ (circles) and $\hat{\varepsilon}_t/\hat{\sigma}_{t,FTSG}$ (crosses) for different cities. From left to right upper panel: Portland, Atlanta, New York, Houston. From left to right lower panel: Berlin, Essen, Tokyo, Osaka, Beijing, Taipei. . . . .	61
4.9	Map of locations where temperature are collected . . . . .	63
4.10	Daily average temperature (blue line) and fourier truncated seasonality function (red line) for Koahsiung. . . . .	63
4.11	Empirical (blue line) and Local linear regression (red line) seasonal variation function for Koahsiung. . . . .	63
4.12	Kernel density estimates for standardized residuals ( $\frac{\hat{\varepsilon}_t}{\hat{\sigma}_{t,LLR}}$ ) for Koahsiung (left panel) and Log densities normal fitting (solid line) and non-parametric fitting (dotted line) (right panel) . . . . .	64
4.13	Localized model selection . . . . .	66
4.14	The Berlin CAT term structure of volatility (black line) and $\sigma_t$ (dash line) from 2004-2008 (left) and 2006 (right) for contracts traded before (upper panel) and within (lower panel) the measurement period. . . . .	72
4.15	Berlin CAT volatility and AR(3) effect of 2 contracts issued on 20060517: one with whole June as measurement period (blue line) and the other one with only the 1st week of June (red line) . . . . .	72
4.16	Two constant MPRs with $\xi = 62, 93, 123, 154$ days for Berlin CAT contracts traded on 20060530. . . . .	77
4.17	Prices (1 panel) and MPR for CAT-Berlin (left side), CAT-Essen (middle side), AAT-Tokyo (right side) of futures traded on 20050530 and 20060531. Constant MPR across contracts per trading day (2 panel), 2 constant per trading day OLS2-MPR (3 panel), time dependent MPR using spline (4 panel). . . . .	78
4.18	Smoothing (black line) 1 day (left), 5 days (middle), 20 days (right) of the MPR parametrization cases (gray crosses) for Berlin CAT Futures traded on 20060530. The last panel gives smoothed MPR estimates for all available contract prices. . . . .	81
4.19	Calibrated MPR and Monthly Temperature Variation of AAT Tokyo Futures from November 2008 to November 2009 (prices for 8 contracts were available). MPR here is a nonmonotone quadratic function of $\hat{\sigma}_{\tau_1, \tau_2}^2$ . . . .	82



*List of Figures*

4.20 Risk premiums (RP) of CAT-Berlin future prices traded during (20031006-20080527) . . . . .	86
---	----

## List of Tables

3.1	Trigger events in earthquakes historical data. . . . .	25
3.2	Confidence Intervals for $\lambda_3$ the intensity rate of events from the earthquake historical data. . . . .	25
3.3	Calibration of intensity rates: the intensity rate $\lambda_1$ from the reinsurance market, the intensity rate $\lambda_2$ from the capital market and the historical intensity rate $\lambda_3$ . . . . .	26
3.4	Cumulative Default Rate comparison from Moody's (Mo) and Standard and Poor's S&P (in % for up to 10 years). . . . .	27
3.5	Descriptive statistics for the variables time $t$ , depth $DE$ , magnitude $Mw$ and loss $X$ of the loss historical data . . . . .	29
3.6	The coefficients of the linear regression loss models and its corresponding coefficients of determination $R^2$ and standard errors $SE$ for the modelled loss data with, without the earthquake in 1985 (EQ-1985) and without the earthquake in 1999 (EQ-1985,1999). . . . .	30
3.7	Parameter estimates by $A^2$ minimization procedure and test statistics for the modelled loss data with and without the 1985 earthquake outlier (EQ-85). In parenthesis, the related $p$ -values based on 1000 simulations. . . . .	33
3.8	Parameter estimates by $A^2$ minimization procedure and test statistics for the earthquake data. In parenthesis, the related $p$ -values based on 1000 simulations. . . . .	36
3.9	Quantiles of 3 years accumulated modelled losses . . . . .	38
3.10	Minimum and maximum of the differences in the (Zero) Coupon CAT bond prices ((Z)CCB) (in % of principal), for the Burr-Pareto distributions of the modelled loss data and the Gamma-Pareto-Weibull distributions of the modelled loss data without the outlier of the earthquake in 1985 (EQ-85). . . . .	40
3.11	Percentages in terms of of the mean of the absolute differences and the mean of the absolute values of the relative differences of the (Zero) Coupon CAT bond prices (Z)CCB for different loss models and the (Z)CCB prices . . . . .	41
4.1	Coefficients of the Fourier truncated seasonal series of average daily temperatures in different cities. All coefficients are nonzero at 1% significance level. Confidence intervals are given in parenthesis. . . . .	51

## List of Tables

4.2	ADF and KPSS-Statistics, coefficients of the autoregressive process $AR(3)$ , $CAR(3)$ and eigenvalues $\lambda_{1,2,3}$ , for the daily average temperatures time series for different cities. +0.01 critical values, * 0.1 critical value (0.11), **0.05 critical value (0.14), ***0.01 critical value (0.21). . . . .	58
4.3	First 7 coefficients $\{c_l\}_{l=1}^7$ of seasonal variance $\sigma_t^2$ fitted with a Fourier truncated series. The coefficients are significant at 1% level. Skewness (Skew), kurtosis (Kurt) and values of Jarque Bera (JB) test statistics of standardized residuals with seasonal variances fitted with GARCH-Fourier series $\hat{\varepsilon}_t/\hat{\sigma}_{t,FTSG}$ and with local linear regression $\hat{\varepsilon}_t/\hat{\sigma}_{t,LLR}$ . Critical value at 5% significance level is 5.99, at 1% is -9.21. . . . .	62
4.4	Wald-stat (WS), Probabilities (Prob), Minimum (Min), Maximum (Max), Median (Med), Standard deviation (Std) of different MPR parametrization (Constant per contract, constant per trading date 'OLS', 2 constant per trading day 'OLS2', Bootstrap and Spline) for CAT-Berlin, CAT-Essen and AAT-Tokyo futures traded during (20031006-20080527), (20050617-20090731) and (20040723-20090630). . . . .	83
4.5	Root mean squared error (RMSE) of CAT, HDD, AAT future observed prices (Bloomberg) and future price estimates from different MPR parametrizations for contracts with $t \leq \tau_1^i < \tau_2^i$ (MPR equal to zero $F_{\hat{\theta}_t=0}$ , constant MPR for different contracts $F_{\hat{\theta}_t^i}$ , constant MPR 'OLS' per trading date $F_{\hat{\theta}_t}$ , 2 constant MPR 'OLS2' $F_{\hat{\theta}_t^{OLS2}}$ , bootstrap MPR $F_{\hat{\theta}_t^{boots}}$ and spline MPR $F_{\hat{\theta}_t^{spl}}$ ). . . . .	85

*Weather has toyed with mankind ever since the first caveman blinked into the baking sun. Some of his descendants eventually became insurance underwriters, offering to ease the sting of droughts and storms for a fee. But only recently has the species worked out how to turn nature -with all its vagaries- into tradable asset in its own right. As the world's grows more volatile, interest in trading is likely to grow, too. Hedge funds, in particular favour the instruments linked to the changeable climate: weather derivatives and catastrophe bond. All are welcome innovations in risk management as insurance and banking increasingly overlap. And all are in growing demand..."*

*The Economist, February 8th 2007.*

# 1 Introduction

*In Nature's Casino.*  
The New York Times, August 26th 2007

Weather influences our daily lives and choices and has an enormous impact on corporate revenues and earnings. The global climate changes the volatility of weather and the occurrence of extreme weather events increases. Adverse and extreme natural events like hurricanes, long cold winters, heat waves, droughts, freezes, etc. may cause substantial financial losses. The traditional way of protection against unpredictable weather conditions has always been the (re)insurance, which covers the loss in exchange for the payment of a premium. However, recently have become popular new financial instruments linked to weather conditions: catastrophe (CAT) bonds and weather derivatives (WD) that are end-products of a process known as securitization that transform non-tradable risk factors (weather or natural catastrophe) into tradable financial assets.

CAT bonds are bonds whose coupons and principal payments depend on the performance of a pool or index of natural catastrophe risks, or on the presence of specified trigger conditions. They cover risk of earthquake, windstorm, hurricane, etc. and transfer the natural risk from insurers, reinsurance and corporations (sponsors) to capital market investors. In case of event, the Special Purpose Vehicle (SPV) gives the principal back to the investors with final coupon; otherwise SPV pays the insured loss and investors sacrifices fully/partially their principal plus interest. For insurers, reinsurers and other corporations, CAT bonds are hedging instruments that offer multi-year protection without the credit risk present in reinsurance by providing full collateral for the risk limits offered through the transaction. For investors CAT bonds offer attractive returns and reduction of portfolio risk, since CAT bonds defaults are uncorrelated to the defaults of other securities. Weather derivatives are financial contracts whose payments are based on weather related measurements and often written on non-tradable underlyings. They differ from insurance by covering lower risk high probability events and payments are made based on various weather elements, such as temperature, precipitation in the form of rain, snow or wind. The Economist date February 10th -16th 2007 estimates further development of these instruments, since they are convenient tools for efficient weather risk management.

The key factor in efficient usage of CAT bonds and WD is a reliable valuation procedure. However, due to their specific nature one encounters several difficulties. Firstly, they are different from most financial derivatives because the underlying weather or natural catastrophe (and indices) are not tradable. Secondly, perfect replication of such products is in general not possible, and therefore we are in the context of incomplete

## 1 Introduction

markets. In other words, since one cannot trade temperature, snowfall, or rainfall, the knack is to hedge a position by trading something that behaves very like it. The pricing and hedging of non-tradable assets can be done in two ways: pricing according to the general equilibrium theory for incomplete markets or pricing via no arbitrage arguments. The first usually requires very detailed assumptions to guarantee appropriate results; the second, although less demanding in terms of assumptions, requires the selection of an adequate equivalent martingale measure to value the payoffs by taking expectations. Since appropriate measures of the risk associated to a particular price become necessary for pricing, one essentially needs to incorporate the market price of risk (MPR), which is an important parameter of the associated equivalent martingale measure. The MPR adjusts the underlying process so that the level of the risk aversion is not needed for valuation. The main goal of this thesis is to discuss the differences between historical and risk neutral behaviors of the non-tradable underlyings and gives insights into the market price of weather risk - MPR (change of drift).

As the study of natural catastrophe models plays an important role in the prevention and mitigation of disasters, the first motivation of this thesis is the analysis of pricing CAT bonds. Baryshnikov et al. [2001] presented an arbitrage-free solution to the pricing of CAT bonds under conditions of continuous trading and according to the statistical characteristics of the dominant underlying processes. Also under an arbitrage-free framework, Vaugirard [2003] evaluated catastrophe bonds by Monte Carlo simulation methods and stochastic interest rates. Burnecki and Kukla [2003] and Burnecki et al. [2005] corrected and applied the results of Baryshnikov et al. [2001] to calculate non-arbitrage prices of a zero coupon and coupon CAT bond. Lee and Yu [2002] developed a methodology that incorporates stochastic interest rates and more generic loss processes to price default-risky CAT bonds. They also analysed the value of the bond under the considerations of default risk, moral hazard and basis risk. Instead of pricing, Anderson et al. [2000] provided benefits to CAT bond by introducing an extensive relative value analysis. Cummins et al. [2004] studied the effectiveness of loss index securities in hedging catastrophic risk. Others, like Croson and Kunreuther [2000] focused on the CAT management and their combination with reinsurance. Lee and Yu [2007] examined how a reinsurance company can increase the value of a reinsurance contract and reduce its default risk by issuing CAT bonds. Barrieu and Loubergé [2009] pointed out that the downside risk aversion and ambiguity aversion have caused the limited success of CAT bonds, therefore they proposed to replace simple CAT bonds with hybrid CAT bonds providing catastrophic risk transfer with protection against a stock crash to complete the market. Cummins and Weiss [2009] and Cummins and Trainor [2009] argued that securitization permits insurers and reinsurers to achieve optimal combination of diversification and shifting of catastrophic risk to the capital markets. The link between parametric/index CAT bonds and reinsurance has been investigated by Finken and Laux [2009], who complained that parametric or index CAT bonds provide low-risk insurers with an alternative to reinsurance contracts, leading to less cross-subsidization in the reinsurance market.

The first part of the thesis examines also the calibration of a real parametric CAT bond for earthquakes and it is of high interest, as it delivers several policy-relevant findings,

## 1 Introduction

e.g. on the relative costs of reinsurance and CAT bonds mixes or the inherent default risk of CAT bonds. The calibration is based on the estimation of the implied trigger intensity rate from the two sides of the contract: from the reinsurance market and from the capital market. A comparative analysis of the risk neutral trigger intensity rates with respect to the historical (physical) intensity rate is conducted to know whether the sponsor company is getting protection at a fair price or whether the CAT bond is sold to the investors for a reasonable price. The results demonstrate that the trigger intensities are lower than the historical one, meaning that the direct access of the CAT bond into Capital Markets expand the risk bearing capacity beyond the limited capital held by reinsurers. Under specific conditions, the financial strategy of the government, a mix of reinsurance and CAT bond is optimal in the sense that it provides coverage for a lower cost and lower exposure at default than the reinsurance itself. For a comparison of the regulation of catastrophic risk financing and other government policies in United States and the European Union, see Klein and Wang [2009].

Since other variables affect the value of the losses, the pricing of a hypothetical CAT bond with a modelled-index loss trigger for earthquakes is also considered. This new approach is also fundamentally driven by the desire to minimise the basis and moral risk borne by the sponsor, while remaining non-indemnity based. The modelled loss is connected with an index CAT bond via the compound doubly stochastic Poisson pricing methodology from Burnecki and Kukla [2003] and Burnecki et al. [2005]. The robustness of the modelled loss with respect to the CAT bond prices is analysed. Because of the quality of the data, the results show that there is no significant impact of the choice of the modelled loss on the CAT bond prices. However, the expected loss is considerably more important for the evaluation of a CAT bond than the entire distribution of losses.

Another motivation of the thesis is to discuss the possibilities of the implementation of financial statistics modelling techniques in the empirical analysis of the weather derivative market. The pricing of weather derivatives has attracted the attention of many researchers. There has been basically two branches of temperature derivative pricing: the indifference pricing approach and pricing models that are based on the evolution of the temperature process. Davis [2001] proposed a marginal utility technique to price temperature derivatives based on the heating degree day (HDD) index. Barrieu and Karoui [2002] presented an optimal design of weather derivatives in an illiquid framework, arguing that the standard risk neutral point of view is not applicable to value them. Richards et al. [2004] applied an extended version of Lucas' (1978) equilibrium pricing model where direct estimation of market price of weather risk is avoided, while Platen and West [2005] used the world stock index as the numeraire to price temperature derivatives. Alaton et al. [2002], Brody et al. [2002] and Benth [2003] fitted an Ornstein-Uhlenbeck stochastic process to temperature data at Chicago O'Hare and Bromma (Stockholm) airport and price futures on temperature indices. Campbell and Diebold [2005] modeled temperature in several US cities with a higher order autoregressive model. They observed seasonal behavior in the autocorrelation function (ACF) of the squared residuals. Mraoua and Bari [2007] studied and priced the temperature in Casablanca, Morocco using a mean reverting model with stochastic volatil-

## 1 Introduction

ity. Similarly Benth and Benth [2007] and Benth et al. [2007b] proposed a continuous autoregressive time model with seasonal variation for the temperature evolution in Stockholm.

The second part of the thesis deals exactly with the differences between historical and risk neutral behaviors of temperature. The majority of papers so far has priced non-tradable assets assuming zero MPR, but this assumption yields biased prices and has never been quantified earlier. The MPR is of high scientific interest, not only for financial risk analysis, but also for better economic modelling of fair valuation of risk. Benth and Benth [2007] introduce theoretical ideas of equivalent changes of measure to get no arbitrage future/option prices written on different indices. Hung-Hsi et al. [2008] examine the effects of mean, variance and market price of risk on temperature option prices and demonstrate that their effects are similar to those on the prices of traditional options. Jewson et al. [2005] argue that the valuation of a WD is equal to the expected outcome under the physical probability plus a charge depending on a risk measure (usually the standard deviation).

Given that liquid derivatives contracts based on daily temperature are traded on the Chicago Mercantile Exchange (CME), the second topic considered in this thesis concerns of inferring the MPR (changes of drift) from traded futures type contracts (CAT, CDD, HDD and AAT) based on a well known pricing model developed by Benth et al. [2007b]. In contrast to this work, the seasonality and seasonal variation of temperature are approximated with a local linear process to get, independently of the chosen location, the driving stochastics close to a Wiener Process and with that being able to work under the financial mathematical context (i.e. an adequate derivative pricing and hedging can be done). A local adaptive modeling approach is proposed to find at each time point an optimal smoothing parameter to locally estimate the seasonal variation.

The implied MPR approach is between a calibration procedure for financial engineering purpose and an economic and statistical testing approach. In the former case, a single date (but different time horizons and calibrated instruments are used) is required, since the model is recalibrated daily to detect intertemporal effects. In the latter case, a specification of the MPR is given and then check consistency with the data. Different specifications of the MPR are investigated. Since smoothing estimates are fundamentally different from estimating a deterministic function, the results are also assure by fitting a parametric function to all available contract prices (calendar year estimation). The empirical results show how the MPR significantly differs from zero, how it varies in time and changes in sign. It is not a reflection of bad model specification, but truly MPR. In particular, the sign changes are determined by the risk attitude and time horizon perspectives of market participants in the diversification process to hedge weather risk and their effect on the demand function. It can be parameterized, given its dependencies on time and temperature seasonal variation and therefore one can infer the MPR for regions without formal weather derivative markets. Connections between the market risk premium (RP) and the MPR are also established. This brings significant challenges to the statistical branch of the pricing literature, suggesting that for regions with homogeneous weather risk there is a common market price of weather risk. With the information extracted we price other degree days futures/options, non-standard



## 1 Introduction

maturity contracts and basket derivatives.

The thesis is organized as follows. The next Chapter is devoted to introduce the financial theory needed in the further chapters. Definitions of the financial market, theory on stochastic integration and differentiation, derivative pricing and the no arbitrage/martingale approach are described. Chapter 3 discusses fundamentals of CAT bonds. The first section - 3.1 - deals with the structure of the CAT bond market, definitions and modelling. Section 3.2 concentrates on the calibration of the real pure parametric CAT bond for earthquakes in Mexico and section 3.3 presents the pricing of a theoretical modelled-index CAT bond fitted to earthquake data in Mexico. This chapter follows the arguments presented in Haerdle and López-Cabrera [2008] and Haerdle and López-Cabrera [2010a].

Chapter 4 focuses on the pricing of weather derivatives. Section 4.1 discusses the fundamentals of temperature index and the monthly temperature futures traded at CME, the biggest market offering this kind of product. In section 4.2 reviews the characteristics of daily average temperatures and the stochastics model that describe the average daily temperature dynamics. At the end of section 4.3, an empirical analysis is conducted to explaining the dynamics of temperature data in different cities. The temperature model captures linear trend, seasonality, mean reversion, long memory (strong autocorrelation) and seasonal volatility effects. Since temperature markets are mean reverted process explained by the conservation of energy, the thesis concentrates on Ornstein-Uhlenbeck (OU) models that model mean reversion in a natural way. Section 4.3 connects the empirical results of the weather dynamics with the pricing methodology. In section 4.4, using real data, we solve the inverse problem of determining the MPR and give (statistical and economic) interpretations of the estimated MPR and risk premia. Section 4.5 shows the relationship between MPR and risk premia. We specify the MPR by introducing a new change measures and give insights of the temperature future curve. Since market participants are affected by weather risk at more than one location, the pricing of future and options written on a basket index of temperatures at several cities is investigated in section 4.6. Section 4.7 concludes the chapter. This chapter follows the arguments presented in Haerdle and López-Cabrera [2010b], Benth et al. [2010] and Haerdle et al. [2010].

All quotations of money in this thesis will be in USD and therefore we will omit the explicit notion of the currency. All the CAT bond computations were carried out in XploRe program version 4.8, while for the weather derivative computations Matlab version 7.6 was used. The earthquake data was provided by the National Institute of Seismology in Mexico, SSN (2006). The temperature data for different cities in US, Europe and Asia was obtained from the National Climatic Data Center (NCDC), the Deutscher Wetterdienst (DWD), Bloomberg Professional Service and the Japanese Meteorological Agency (JMA). The Weather Derivative data was provided by Bloomberg Professional service.

## 2 Theoretical Background

*Global warming and Financial Umbrellas.*  
Dosi and Moretto [2003]

In the next section the different modelling issues are discussed to establish a link between the theoretical and empirical analysis which will be our focus in the subsequent chapters. In order to understand these models, we need to understand the theory on stochastic integration and differentiation. For this thesis, the most important results will be the Itô's Formula, the stochastic Fubini Theorem, the Bayes' Formula and Lévy-Kintchine representation for a deeper discussion see Musiela and Rutkowski [1997], Shreve [2004] and Dupacová et al. [2002]. The chapter includes some of the most used stochastic process in Weather Markets.

In section 2.2 both geometric and arithmetic models based on Ornstein-Uhlenbeck (OU) processes are analysed to model the mean reversion of the underlying in a natural way. The future price dynamics are studied in Section 2.3. Here the Esscher transformation plays an important rule since it does not only preserve the distributional properties of the jump processes, but constructs risk neutral probabilities. The Esscher transform can be seen as a generalization of the Girsanov transform used for Brownian motions. The temperature price dynamics and the market price of risk are studied in Chapter 4.

### 2.1 Stochastic Analysis

Let  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$  be a complete filtration probability space. A random variable is a mapping  $X: \Omega \mapsto \mathbb{R}^d$  if it is  $\mathcal{F}$ -measurable, whereas a family of random variables depending on time  $t$ ,  $\{X_t\}_{t \geq 0}^T$  is said to be a stochastic process. A process  $X_t$  is  $\mathcal{F}$ -adapted if every  $X_t$  is measurable with respect to the  $\sigma$ -algebra  $\mathcal{F}_t$ . If the paths  $t \mapsto X_{(t,w)}$  are right continuous with left limits everywhere with probability one, then the stochastic process is called càdlàg.

A stopping time  $\tau$  is a random variable with values in  $[0, \infty]$  and with the property  $\{w \in \Omega | \tau(w) \leq t\} \in \mathcal{F}_t$ , for every  $t \geq 0$ . An adapted càdlàg stochastic process  $M_t$  is called a martingale if it is in  $L^1(P)$  for all  $t \geq 0$  and for every  $t \geq s \geq 0$ :

$$E[M_t | \mathcal{F}_s] = M_s \quad (2.1)$$

$M_t$  is a local martingale if there exists a sequence of stopping times  $\tau_n < \infty$  where  $\tau_n \uparrow \infty$ , such that  $M_{t \wedge \tau_n}$  is a martingale.

Consider a finite time horizon  $[0, T]$  and let  $Q$  be a probability measure equivalent

## 2 Theoretical Background

to  $P$ . Let  $Z_t$  be the density process of the Random Nikodym derivative so that:

$$Z_t = \frac{dQ}{dP} \Big|_{\mathcal{F}_t} \quad (2.2)$$

Suppose that the  $Z_t$  is a martingale, then the conditional expectations with respect to different probabilities can be calculated with the Bayes' Formula:

$$E[X|\mathcal{F}_t] = Z_t^{-1} E[XZ_T|\mathcal{F}_t] \quad (2.3)$$

where  $X$  is a integrable random variable and  $E[\cdot]$  is the expectation operator with respect to  $Q$ , Liptser and Shiryaev [1991].

An adapted càdlàg stochastic process  $I_t$  starting at zero is an II process if:

1. For any partition  $0 \leq t_0 < t_1 < \dots < t_n$  for  $n \geq 1$ , the increments  $I_{t_0}, I_{t_1} - I_{t_0}, \dots, I_{t_n} - I_{t_{n-1}}$  are independent random variables.
2. For every  $t \geq 0$  and  $\epsilon > 0$ , it is continuous in probability  $\lim_{s \rightarrow t} P(|I_s - I_t| \geq \epsilon) = 0$ .

Additionally, if the increments are stationary, in the sense that the distribution of  $I_t - I_s$  is dependent on the interval  $t - s$ , not on  $s$  and  $t$  separately, with  $0 \leq s < t$ , then the stochastic process  $I_t$  is called a Lévy process ( $L_t$ ). When the increment  $L_t - L_s$  is normal distributed with zero mean and variance  $t - s$ , the process is called a Brownian motion  $B_t$ .

For  $0 \leq s < t, \theta \in \mathbb{R}$ , the characteristic function of the II process (cumulant function of  $I_t$ ) is:

$$E[\exp\{i\theta(I_s - I_t)\}] = \exp\left\{\psi_{(s,t,\theta)}\right\} \quad (2.4)$$

$$\begin{aligned} \psi_{(s,t,\theta)} &= i\theta(\gamma_s - \gamma_t) - \frac{1}{2}\theta^2\{C_t - C_s\} \\ &+ \int_s^t \int_{\mathbb{R}} \{\exp(i\theta z) - 1 - i\theta z \mathbf{1}(|z| \leq 1)\} \ell(dz, du) \end{aligned} \quad (2.5)$$

where  $\gamma : \mathbb{R} \mapsto \mathbb{R}, \gamma_0 = 0$ ,  $C : \mathbb{R} \mapsto \mathbb{R}, C_0 = 0$  nondecreasing and both functions being continuous. The compensator measure  $\ell$  relates to the jumps of the II process and is a  $\sigma$ -finite measure on the Borel  $\sigma$ -algebra of  $[0, \infty) \times \mathcal{B}$  with the properties  $\ell(A \times 0) = 0, \ell(t \times \mathbb{R}) = 0, \int_0^t \int_{\mathbb{R}} \min(1, z^2) \ell(ds, dz) < \infty, t \geq 0$  and  $A \in \mathcal{B}(\mathbb{R}_+)$ , Ikeda and Watanabe [1981].

Let  $\mathcal{M}_2$  be the set of martingales  $M_t$  that are square integrable. The Doob-Meyer decomposition theorem states that if  $M_t \in \mathcal{M}_2$  then there exist a unique natural increasing process  $A_t$  such that  $M_t^2 - A_t$  is a martingale. If  $M_t, N_t \in \mathcal{M}_2$  then there exists a unique process  $A_t$  such that  $M_t N_t - A_t$  is also a martingale. The process  $A_t$  is usually called Quadratic variation process of the martingale  $M_t$  and it is denoted as  $\langle M, M \rangle$  or quadratic variation process of the martingales  $M_t$  and  $N_t$  as  $\langle M, N \rangle$ .

## 2 Theoretical Background

The class of square integrable martingales are suitable as stochastic integrators. We say that the stochastic process  $X_t$  is integrable with respect to  $\mathcal{M}_2$ , if:

1.  $X_t$  is predictable:  $X_t$  is measurable with respect to the smallest  $\sigma$ -algebra on  $[0, \infty) \times \Omega$  such that all left continuous processes are measurable.
2.  $E \left[ \int_0^t X_s^2 d\langle M \rangle_s \right] < \infty$ , for every  $t > 0$ . The Itô isometry for stochastic integrals with respect to Brownian Motion is defined as:

$$E \left[ \int_0^t X_s^2 d\langle M \rangle_s \right] = E \left[ \left( \int_0^t X_s dM_s \right)^2 \right]$$

II process are closely related to semimartingales, which are very tractable tools for analysis since they are closed under stochastic integrations, differentiation (Itô's Formula) and measure change, among other things. The Lévy-Kintchine decomposition of  $I_t$  gives the connection to semimartingales:

$$I_t = \gamma_t + M_t + \int_0^t \int_{|z|<1} z \tilde{N}_i(ds, dz) + \int_0^t \int_{|z|>1} z N_j(ds, dz) \quad (2.6)$$

where  $M_t$  is a local square integrable continuous martingale with quadratic variation equal to  $C_t$ .  $N$  denotes the random jump measure associated to the II process,  $\tilde{N} = N - \ell$  stands for the compensated random jump measure.

Then an adapted càdlàg stochastic process  $S_t$  is a semimartingale if it has the Lévy-Kintchine representation

$$S_t = S_0 + A_t + M_t + \int_0^t \int_{\mathbb{R} \setminus \{0\}} X_{1(t,z)} \tilde{N}(ds, dz) + \int_0^t \int_{\mathbb{R} \setminus \{0\}} X_{2(t,z)} N(ds, dz) \quad (2.7)$$

where  $A_t$  is an adapted continuous stochastic process having paths of finite variation on finite time intervals,  $M_t$  is a continuous square integrable local martingale,  $S_0$  is an  $\mathcal{F}_0$ -measurable random variable,  $X_{1(t,z,w)}, X_{2(t,z,w)}$  are predictable random variables defined on  $[0, \infty) \times \mathbb{R} \times \Omega$  with  $X_{1(t,z,q)} X_{2(t,z,q)} = 0$ , satisfying

$$E \left[ \int_0^t \int_{\mathbb{R} \setminus \{0\}} |X_{(s,z)}|^2 \ell(ds, dz) \right] < \infty$$

and

$$\int_0^t \int_{\mathbb{R} \setminus \{0\}} |X_{(s,z)}| N(ds, dz) < \infty, a.s$$

The stochastic Fubini Theorem, Protter [1990], says that for a semimartingale  $X_t$ ,  $(U, \mathcal{U})$  a measure space equipped with a finite measure  $m(du)$ , a predictable  $\sigma$ -algebra  $\mathcal{P}$  and  $H_{(u,t,w)}$  be a  $\mathcal{U} \otimes \mathcal{P}$  measurable, with  $\left\{ \int_u H_{(u,t,\cdot)}^2 m(du) \right\}^{1/2}$  assumed to be integrable with respect to  $X_t$  and letting  $\int_0^t H_{(u,s,\cdot)} dX_s$  be a  $\mathcal{U} \otimes \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F}$  measurable and

## 2 Theoretical Background

càdlàg for each  $u$ , then

$$\int_U \int_0^t H_{(u,s,\cdot)} dX_s m(du)$$

exists and is a càdlàg version of

$$\int_0^t \int_U H_{(u,s,\cdot)} m(du) dX_s$$

In the next chapters,  $U$  will be an interval in  $\mathbb{R}_+$ , with  $\mathcal{U}$  being the Borel set and  $m(du)$  the Lebesgue measure.

Based on the multi-dimensional formula in Protter [1990] and the fact that jumps  $N_i$  are independent for the semimartingale process, the Itô Formula for the set  $\mathbf{S}_t = \{S_{1(t)}, \dots, S_{n(t)}\}$  of  $n$  semimartingales, with dynamics defined in (2.7) is equal to:

$$\begin{aligned} S_{t(i)} &= S_{0(i)} + A_{t(i)} + M_{t(i)} + \int_0^t \int_{\mathbb{R} \setminus \{0\}} X_{1,i(t,z)} \tilde{N}_i(ds, dz) \\ &\quad + \int_0^t \int_{\mathbb{R} \setminus \{0\}} X_{2,i(t,z)} N_i(ds, dz) \end{aligned} \quad (2.8)$$

Further, let  $f_{(t,x)}$  be a real valued function on  $[0, \infty) \times \mathbb{R}^n$ , once continuously differentiable in  $t$  and twice continuously differentiable in  $x$ . Then  $f_{(t,\mathbf{S}_t)}$  is again a semimartingale, with the following representation:

$$\begin{aligned} f_{(t,\mathbf{S}_t)} &= f_{(0,\mathbf{S}_0)} + \int_0^t \partial_t f_{(u,\mathbf{S}_u)} du + \sum_{i=1}^n \int_0^t \partial_{x_i} f_{(u,\mathbf{S}_u)} dA_{i(u)} + \sum_{i=1}^n \int_0^t \partial_{x_i} f_{(u,\mathbf{S}_u)} dM_{i(u)} \\ &\quad + \frac{1}{2} \sum_{i,j=1}^n \int_0^t \partial_{x_i x_j} f_{(u,\mathbf{S}_u)} d\langle M_i, M_j \rangle_u \\ &\quad + \sum_{i=1}^n \int_0^t \int_{\mathbb{R} \setminus \{0\}} f_{\{u,\mathbf{S}_{u-} + X_{1,i(u,z)} \mathbf{e}_i\}} - f_{(u,\mathbf{S}_{u-})} \tilde{N}_i(du, dz) \\ &\quad + \sum_{i=1}^n \int_0^t \int_{\mathbb{R} \setminus \{0\}} f_{\{u,\mathbf{S}_{u-} + X_{2,i(u,z)} \mathbf{e}_i\}} - f_{(u,\mathbf{S}_{u-})} N_i(du, dz) \\ &\quad + \sum_{i=1}^n \int_0^t \int_{\mathbb{R} \setminus \{0\}} f_{\{u,\mathbf{S}_u + X_{1,i(u,z)} \mathbf{e}_i\}} - f_{(u,\mathbf{S}_u)} - X_{1,i(u,z)} \partial_{x_i} f_{(u,\mathbf{S}_u)} \ell_i(du, dz) \end{aligned} \quad (2.9)$$

where  $\partial_t f$  and  $\partial_{x_i} f$  are the first derivatives with respect to  $t$  and  $x_i$  of  $f(t, x)$  and  $\partial_{x_i x_j} f$  is the second derivative of  $f$  with respect to  $x_i$  and  $x_j$ ,  $\mathbf{e}_i, i = 1 \dots n$  is the  $i$ th basis vector in  $\mathbb{R}^n$ , with 1 on coordinate  $i$  and zeros otherwise.

A Poisson process  $N_t$  with intensity  $\lambda$  is a one dimensional stochastic process which has stationary and independent increments,  $N_t - N_s$  is Poisson distributed with inten-

## 2 Theoretical Background

sity  $\lambda(t-s), 0 \leq s < t$ :

$$P(N_t - N_s = k) = \frac{\lambda^k (t-s)^k}{k!} \exp \{-\lambda(t-s)\} \quad (2.10)$$

A Poisson process jumps with size one at exponentially distributed jumping times, and remain constant between jumps. A Compound Poisson process  $L_t$  is Lévy process, where jumps occur at an intensity  $\lambda$ , and jump size is defined by i.i.d random variables  $X_i$ :

$$L_t = \sum_{i=1}^{N_t} X_i \quad (2.11)$$

where  $N_t$  is a Poisson process with intensity  $\lambda$  independent of  $X_i$ .

For  $t \leq s \leq \mathcal{T}$ , a càdlàg process

$$\begin{aligned} X_s = & x \exp \left( - \int_t^s \alpha_v dv \right) + \int_t^s \mu_u \exp \left( - \int_t^s \alpha_v dv \right) du \\ & + \int_t^s \sigma_u \exp \left( - \int_t^s \alpha_v dv \right) dB_u \end{aligned} \quad (2.12)$$

is an Ornstein-Uhlenbeck (OU) process if it is the unique solution of the stochastic differential equation:

$$dX_s = (\mu_s - \sigma_s X_s) ds + \sigma_s dB_s, X_t = x \quad (2.13)$$

The proof is based on Itô's Formula, see Ikeda and Watanabe [1981].

### 2.2 Stochastic price modelling

A dynamic of the spot price evolution is desirable, firstly because modelling uncertainty in spot prices is of interest for traders, and secondly because spot prices are used as reference point for settlement of forward and future contracts.

In mathematical finance, the traditional models are based on Brownian Motion  $B_t$ , also called Wiener Process. The most common model for the price dynamics  $S_t$  of a financial asset is the exponential of a drift Brownian motion, known as the geometric Brownian motion, Samuelson [1965]:

$$S_t = S_0 \exp (\mu t + \sigma B_t) \quad (2.14)$$

with  $\mu$  and  $\sigma > 0$  being constants. Consequently, the logreturns change over a time interval  $\Delta t$  ( $\ln S_{(t+\Delta t)} - \ln S_t$ ) and become independent, stationary and normally distributed, an important property in view of the market efficiency hypothesis Fama [1970]. A generalization of the geometric Brownian motion is the exponential of a Lévy process  $L_t$  that models jumps and the leptokurtotic behaviour of asset prices on small

## 2 Theoretical Background

time scales:

$$S_t = S_0 \exp(L_t) \quad (2.15)$$

An extension of the geometric Brownian motion allowing for mean reversion is defined in Schwartz [1997]:

$$S_t = S_0 \exp(X_t) \quad (2.16)$$

where  $dX_t = \alpha(\mu - X_t)dt + \sigma dB_t$ . All these models capture price fluctuations and large variations, but the variation in prices is homogeneous over the year. Weather markets are seasonal varying, meaning that the temperature dynamics have seasonal features like time-dependent volatility. The II process opens up for multi-factor models of the Schwartz type which may have factor seasonal dependent jump frequencies and sizes in addition to mean reversion. This class of models is reasonable compromise between modelling flexibility and analytical tractability for computing derivative prices.

Arithmetic models are considered rather than geometrical ones for the spot price evolution since they assume the spot price dynamic as a sum of Ornstein-Uhlenbeck (OU) processes driven by II type process that model mean reversion in a natural way, allow for jumps and seasonal variations. This makes the analytical pricing of future contracts feasible for a large class of models.

Let the stochastic process  $S_t$  be a semimartingale and defined as

$$S_t = \Lambda_t + \sum_{i=1}^m X_{i(t)} + \sum_{j=1}^n Y_{j(t)} \quad (2.17)$$

with

$$\begin{aligned} dX_{i(t)} &= \left\{ \mu_{i(t)} - \alpha_{i(t)} X_{i(t)} \right\} dt + \sum_{k=1}^p \sigma_{ik(t)} dB_{k(t)} \\ dY_{j(t)} &= \left\{ \delta_{j(t)} - \beta_{j(t)} Y_{j(t)} \right\} dt + \eta_{j(t)} dI_{j(t)} \end{aligned} \quad (2.18)$$

where  $\Lambda_t$  is a deterministic continuously differentiable function modelling the seasonal varying mean level, the coefficients  $\mu_i, \alpha_i, \delta_i, \beta_i, \sigma_{ik}$  and  $\eta_j$  are all continuous functions,  $B_k, k = 1 \dots p$  independent Brownian motions and  $n$  pure jump semimartingales II process  $I_{j(t)}, j = 1 \dots n$ , where  $I_{j(t)}$  and  $I_{k(t)}$  are independent for each other and  $j \neq k$ . The random jump measure  $N_j(dt, dz)$  is given by the Lévy-Kintchine representation:

$$I_{j(t)} = \gamma_{j(t)} + \int_0^t \int_{|z| < 1} z \tilde{N}_j(dz, du) + \int_0^t \int_{|z| > 1} z N_j(dz, du) \quad (2.19)$$

where  $\gamma_i$  has a bounded variation and the compensator measure is denoted by  $\ell_j(dz, du)$ . From a modelling point of view, it is reasonable to assume the OU process mean reverting with constant speeds and choose  $\mu_{i(t)} = \delta_{j(t)} = 0$  in order to have the

## 2 Theoretical Background

seasonality function as the mean price level. The initial condition of  $X_i$  and  $Y_i$  is:

$$S_0 - \Lambda_0 = \sum_{i=1}^m X_{i(0)} + \sum_{j=1}^n Y_{j(0)} \quad (2.20)$$

By using (2.12), the explicit representations of  $X_i$  and  $Y_i$  are:

$$\begin{aligned} X_{i(t)} &= X_{i(0)} \exp \left\{ - \int_0^t \alpha_{i(v)} dv \right\} + \int_0^t \mu_{i(u)} \exp \left\{ - \int_u^t \alpha_{i(v)} dv \right\} du \\ &\quad + \sum_{k=1}^p \int_0^t \sigma_{ik(u)} \exp \left\{ - \int_u^t \alpha_{i(v)} dv \right\} dB_{k(u)} \\ Y_{i(t)} &= Y_{i(0)} \exp \left\{ - \int_0^t \beta_{i(v)} dv \right\} + \int_0^t \delta_{j(u)} \exp \left\{ - \int_u^t \beta_{j(v)} dv \right\} du \\ &\quad + \int_0^t \eta_{j(u)} \exp \left\{ - \int_u^t \beta_{j(v)} dv \right\} dI_{j(u)} \end{aligned} \quad (2.21)$$

To ensure that the spot price processes has finite moments up to certain orders is necessary that in the jump process there exists a constant  $c_j > 0, j = 1 \dots n$  in a finite time horizon  $\mathcal{T} < \infty$  such that:

$$\int_0^{\mathcal{T}} \int_{|z|>1} |z|^{c_j} \ell_j(du, dz) < \infty \quad (2.22)$$

Arithmetic models might lead to negative prices. The probability that  $S_t < 0$  will depend on the volatility  $\sigma$ , mean reversion  $\alpha$ , mean level  $\Lambda$  and the size and frequency of jumps. For simplicity, consider the  $S_t$  in (2.17) without jumps ( $n = 0$ ). From (2.21), one can derive the probability that  $S_t < 0$  driven by this Gaussian OU process:

$$P(S_t < 0) = \Phi \left( - \frac{m_t}{\Sigma_t} \right) \quad (2.23)$$

where

$$\begin{aligned} m_t &= \Lambda_t + \sum_{i=1}^m X_{i(0)} \exp \left\{ - \int_0^t \alpha_{i(s)} ds \right\} \\ \Sigma_t^2 &= \sum_{k=1}^p \int_0^t \sigma_{ik(s)}^2 \exp \left\{ - 2 \int_0^t \alpha_{i(u)} du \right\} ds \end{aligned} \quad (2.24)$$

and  $\Phi$  is the cdf of a standard normal distribution. Including jumps to the process will increase or decrease the probability.



### 2.3 Pricing futures on the spot market

The next modelling point is to connect the spot and future price dynamics. The typical models in mathematical finance belong to the class of semimartingale processes. They assume the existence of equivalent (local) martingale measures to the objective (physical or market)  $P$ . These probabilities lead to markets where there are not arbitrage possibilities since the martingale property of the discounted prices makes zero profit.

Since the future contracts need to have a price dynamics being arbitrage free, the future contracts must be adapted to the filtration information set at  $t$ ,  $\mathcal{F}_t$ . Thus for a stochastic process  $S_t$  defining the spot price dynamics defined in the previous section and  $r$  be the constant risk free interest rate, the future price of no-restorable assets with delivery the spot at time  $\tau$  is expressed as:

$$\exp \{-r(\tau - t)\} E^Q [S_\tau - F_{(t,\tau)} | \mathcal{F}_t] = 0 \quad (2.25)$$

Then the formula spot-future relationship is:

$$F_{(t,\tau)} = E^Q [S_\tau | \mathcal{F}_t] \quad (2.26)$$

Assuming that the spot market is complete and liquid and since  $Q$  is a risk neutral probability, it holds that:

$$F_{(t,\tau)} = S_t \exp \{-r(\tau - t)\} \quad (2.27)$$

The rational expectation hypothesis (Samuelson [1965]) assumes that  $Q = P$  and therefore

$$F_{(t,\tau)} = E [S_\tau | \mathcal{F}_t] \quad (2.28)$$

In reality this equality does not hold. The risk premium (RP) measures exactly the difference between the risk neutral probability  $Q$  and the market probability  $P$ :

$$RP_{(t,\tau_1,\tau_2)} = E^Q [S_\tau | \mathcal{F}_t] - E [S_\tau | \mathcal{F}_t] \quad (2.29)$$

The choice of  $Q$  determines the risk premium, and opposite, having knowledge of the risk premium determines the choice of the risk neutral probability. To explain the risk premium, risk neutral probabilities  $Q$  introduce a parametric change of the drift of the spot. Thanks to Girsanov transformation (Bjork [1998]), a change of measure can be made. For a constant market price of risk  $\theta$ , there exists a probability  $Q$  equivalent to  $P$  so that:

$$B_t^\theta = B_t - \theta t \quad (2.30)$$

is a Brownian motion under  $Q$ . In other words, the market price of risk is the return in excess of the risk-free rate that the market wants as compensation for taking risk.

The Esscher Transform is a generalization of the Girsanov transformation for Brow-

## 2 Theoretical Background

nian motions to a general II process. It changes the probability measure by parametrizing drift, jump frequency and size of the spot dynamics, but preserves the independent property. The Esscher Transform restricts the space of potential pricing measures to a subclass of parametrized equivalent measures, see Esscher [1932]. Note that these time-dependent parameters (market price of risks) can be estimated from option/future data and will be related to the risk premium, see Gerber and Shiu [1994].

The Esscher Transform says that if there is an exponential moment  $\int_{\mathbb{R}} \exp(\theta y) f_y dy$ , a new probability density can be defined as

$$f_{(x;\theta)} = \frac{\exp(\theta x) f_x}{\int_{\mathbb{R}} \exp(\theta y) f_y dy} \quad (2.31)$$

where  $f$  is a probability density and  $\theta$  a real number. The transform applied to II process including time-dependent parameter  $\theta_t$ , can be found in Benth et al. [2008]. Let  $\theta_t = \{\hat{\theta}_{1(t)}, \dots, \hat{\theta}_{p(t)}, \tilde{\theta}_{1(t)}, \dots, \tilde{\theta}_{n(t)}\}$  be a  $(p+n)$ -dimensional vector of real-valued continuous functions on  $[0, \mathcal{T}]$ . For  $0 \leq t < \tau$  define the stochastic exponential:

$$\begin{aligned} Z_t^\theta &= \exp \left\{ \int_0^t \hat{\theta}_{k(s)} dB_{k(s)} - \frac{1}{2} \int_0^t \hat{\theta}_{k(s)}^2 ds \right\} \\ &\quad \times \exp \left\{ \int_0^t \tilde{\theta}_{j(s)} dI_{j(s)} - \psi_{j(0,t,\tilde{\theta}_{j(\cdot)})} \right\} \\ &= \prod_{k=1}^p \hat{Z}_{k(t)}^\theta \times \prod_{j=1}^n \tilde{Z}_{j(t)} \end{aligned} \quad (2.32)$$

where  $\psi_{j(0,t,\tilde{\theta}_{j(\cdot)})}$  is the cumulant function of  $I_t$ , (2.5). From Novikov condition, Karatzas and Shreve [1991],  $\hat{Z}_{k(t)}^\theta$  is a martingale since  $\hat{\theta}_k(s)$  is continuous, and from Itô's Formula, it follows that  $\tilde{Z}_{j(t)}$  is also martingale with expectation one. In this case the Radon-Nikodym derivative is defined as:

$$\frac{dQ}{dP} \Big|_{\mathcal{F}_t} = Z_t^\theta \quad (2.33)$$

The  $\hat{\theta}_k$ 's denote the price that market participants give to the risks that cannot be hedged and  $\tilde{\theta}_j$ 's are the market prices for the jump risks. Thus for a market price of risk  $\hat{\theta}_k$ , there exists a probability  $Q_\theta$  equivalent to  $P$  so that:

$$B_{k(t)}^\theta = B_{k(t)} - \int_0^t \hat{\theta}_{k(u)} du \quad (2.34)$$

are Brownian motion under  $Q_\theta$ , and  $I_t$  is an II process (see (2.6)) with drift:

$$\gamma_{j(t)} + \int_0^t \int_{|z|<1} z \left[ \exp \left\{ \tilde{\theta}_{j(u)} z \right\} - 1 \right] \ell_j(dz, du) \quad (2.35)$$

## 2 Theoretical Background

and compensator measure  $\exp \left\{ \tilde{\theta}_{j(t)} z \right\} \ell(dz, dt)$ , Benth et al. [2008]. Suppose that the model in (2.17) holds, then the future price  $F_{(t,\tau)}$  is given as:

$$\begin{aligned}
E^{Q_\theta} [S_\tau | \mathcal{F}_t] &= \Lambda_\tau + E^{Q_\theta} [X_\tau | \mathcal{F}_t] + E^{Q_\theta} [Y_\tau | \mathcal{F}_t] \\
&= \Lambda_\tau + \sum_{i=1}^m X_{i(t)} \exp \left\{ - \int_t^\tau \alpha_{i(s)} ds \right\} + \sum_{j=1}^n Y_{j(t)} \exp \left\{ - \int_t^\tau \beta_{j(s)} ds \right\} \\
&\quad + \sum_{i=1}^m \int_t^\tau \mu_{i(u)} \exp \left\{ - \int_u^\tau \alpha_{i(v)} dv \right\} du \\
&\quad + \sum_{j=1}^n \int_t^\tau \delta_{j(u)} \exp \left\{ - \int_u^\tau \beta_{j(v)} dv \right\} du \\
&\quad + \sum_{k=1}^p \sum_{i=1}^m \int_t^\tau \sigma_{ik(u)} \hat{\theta}_{k(u)} \exp \left\{ - \int_u^\tau \alpha_{i(v)} dv \right\} du \\
&\quad + \sum_{j=1}^n \int_t^\tau \eta_{j(u)} \exp \left\{ - \int_u^\tau \beta_{j(v)} dv \right\} d\gamma_{j(u)} \\
&\quad + \sum_{j=1}^n \int_t^\tau \int_{\mathbb{R}} \eta_{j(u)} \exp \left\{ - \int_u^\tau \beta_{j(v)} dv \right\} \\
&\quad \times z \left[ \exp \left\{ \tilde{\theta}_{j(u)} z \right\} - \mathbf{1}(|z| \leq 1) \right] \ell_j(dz, du)
\end{aligned} \tag{2.36}$$

by (2.21), (2.34) and (2.35).

Alternatively to explaining future prices by the underlying spot with a specification of the risk neutral probability (the market price of risk), one may use the Heath-Jarrow-Morton (HJM) approach from interest rate theory, Heath [1992] which suggest to directly assume dynamics for the future price evolution. This can be done in terms of market dynamics or risk neutral measure. Then a natural way to estimate the market price of risk is by minimising the distance between the theoretical and observed price dynamics after an Esscher transformation. Once having these risk neutral dynamics, one can calculate the conditional expectation of the pay-off from the option and hedging can be done.

Before continuing to the next section, let us discuss the measure change under the Black-Schole framework. Consider two probabilities measures  $P$  and  $Q$ . Assume that  $\frac{dQ}{dP} |_{\mathcal{F}_s} = Z_t > 0$  is a positive Martingale. By Itô's Lemma, then:

$$\begin{aligned}
Z_t &= \exp \{ \log(Z_t) \} \\
&= \exp \left\{ \int_0^t (Z_s)^{-1} dZ_s - \frac{1}{2} \int_0^t (Z_s)^{-2} d\langle Z \rangle_s \right\}
\end{aligned} \tag{2.37}$$

## 2 Theoretical Background

Let  $B_t, Z_t$  be Martingales under  $P$ , then by Girsanov theorem:

$$\begin{aligned}
 B_t^\theta &= B_t - \int_0^t (Z_s)^{-1} d \langle Z, B \rangle_s \\
 &= B_t - \int_0^t (Z_s)^{-1} d \langle \int_0^s \theta_u Z_u dB_u, B_s \rangle \\
 &= B_t - \int_0^t (Z_s)^{-1} \theta_s Z_s ds \\
 &= B_t - \int_0^t \theta_s ds
 \end{aligned} \tag{2.38}$$

is a Martingale under  $Q$ . Then, every transformation of the probability measure has the form of a positive Martingale:

$$Z_t = \exp \left( \int_0^t \theta_s dB_s - \frac{1}{2} \int_0^t \theta_s^2 ds \right) \tag{2.39}$$

where  $dZ_s = Z_s \cdot \theta_s \cdot dB_s$ .

In the Black-Scholes framework, the asset price follows:

$$dS_t = \mu S_t dt + \sigma_t S_t dB_t \tag{2.40}$$

with explicit dynamics:

$$S_t = S_0 + \int_0^t \mu S_s ds + \int_0^t \sigma_s S_s dB_s \tag{2.41}$$

Note that the stochastic process  $S_t$  is not a Martingale under  $P$ , but it is under  $Q$ . By the no arbitrage condition, the risk free interest rate  $r$  should be equal to the drift  $\mu + \theta_s \sigma_s$ , so that:

$$\theta_s = \frac{r - \mu}{\sigma_s} \tag{2.42}$$

In practice  $B_t^\theta = B_t - \int_0^t \left( \frac{\mu - r}{\sigma_s} \right) ds$  is a Martingale under  $Q$  and then  $e^{-rt} S_t$  is also a Martingale. Then the dynamics in (2.41) become

$$\begin{aligned}
 S_t &= S_0 + \int_0^t \mu S_s ds + \int_0^t \sigma_s S_s dB_s^\theta + \int_0^t \theta_s \sigma_s S_s ds \\
 &= S_0 + \int_0^t S_s (\mu + \theta_s \sigma_s) ds + \int_0^t \sigma_s S_s dB_s^\theta
 \end{aligned} \tag{2.43}$$

### 3 Catastrophe (CAT) Bonds

*A brave bet on natural disasters.*  
The Economist, August 2nd 2007

During 1994 through 2004, the market was experimenting with different financial instruments. Many of them did not succeed. The Chicago Board of Trade (CBOT) introduced insurance futures in December 1992, but due to a lack of interest from insurers they were replaced by CBOT catastrophe loss indices options. Later on Act of God bonds were developed to allow insurers to borrow in case of a catastrophic loss. Act of God failed cause insurers were obligated to repay the trust. In the mid-1990's catastrophe bonds (CAT bonds), also named as Insurance linked bond (ILB) and described in more detail below, were developed to ease the transfer of catastrophe based insurance risk from insurers, reinsurers and corporations (sponsors) to capital market investors. CAT bonds releases funds to insurers following catastrophes without creating the obligation to repay. They are formally traded at the Chicago Board of Trade and the first successful Cat bond was issued by Hannover Re in 1994, SwissRe [2001].

The catastrophe bond sector ended 2009 with 3.4 billion of new issuance and it is expected it to climb to 5 USD billion in 2010, (Bloomberg, 2010). The CAT issuance for U.S. perils represented 80% of the total ILS market, followed by issuance for Japan and European perils, and money market funds that are collaterals offering conservative structures and are less dependent on counterparties' credit ratings, were the most popular form of collateral management for ILS.

This chapter is structured as follows. The next section is devoted to definitions of CAT bonds and the trigger mechanisms. Section 3.2 examines the calibration of a real pure parametric CAT bond for earthquakes in Mexico. Section 3.3 presents the pricing of a theoretical modelled-index CAT bond fitted to earthquake data in Mexico and section 3.4 summarises the chapter. Section 3.5 concludes the chapter. This chapter follows the arguments presented in Haerdle and López-Cabrera [2008] and Haerdle and López-Cabrera [2010a].

#### 3.1 Definitions

CAT bonds are bonds whose coupons and principal payments depend on the performance of a pool or index of natural catastrophe risks, or on the presence of specified trigger conditions. They protect sponsor companies from financial losses caused by large natural disasters by offering an alternative or complement to traditional reinsurance.

### 3 Catastrophe (CAT) Bonds

The transaction involves four parties: the sponsor or ceding company (government agencies, insurers, reinsurers), the special purpose vehicle SPV (or issuer), the collateral and the investors (institutional investors, insurers, reinsurers, and hedge funds). The basic structure is shown in Figure 3.1. The sponsor sets up a SPV as an issuer of the bond and a source of reinsurance protection. The issuer sells bonds to capital market investors and the proceeds are deposited in a collateral account, in which earnings from assets are collected and from which a floating rate is paid to the SPV. The sponsor enters into a reinsurance or derivative contract with the issuer and pays him a premium. The SPV usually gives quarterly coupon payments to the investors. The premium and the investment bond proceeds that the SPV received from the collateral are a source of interest or coupons paid to investors. If there is no trigger event during the life of the bonds, the SPV gives the principal back to the investors with the final coupon or the generous interest; otherwise the SPV pays the ceding according to the terms of the reinsurance contract and sometimes pays nothing or partially the principal and interest to the investors.

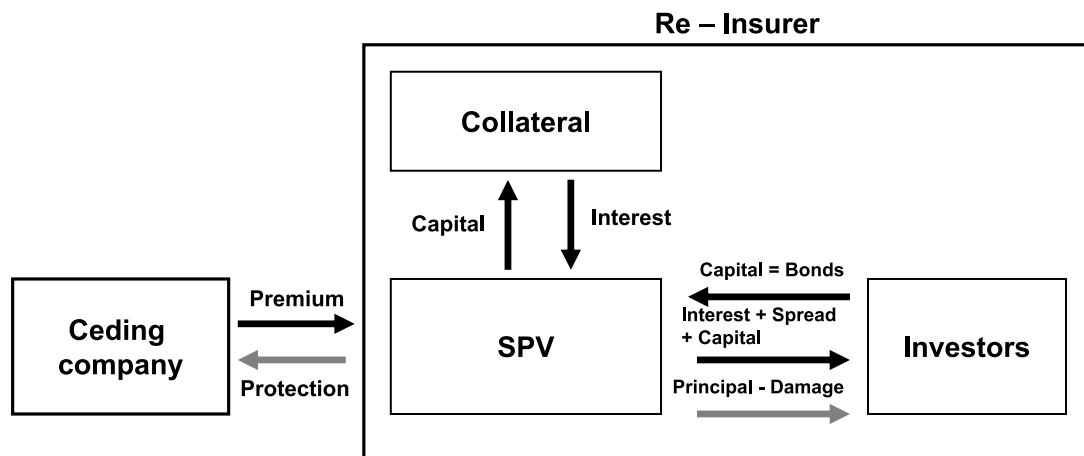


Figure 3.1: Cash flows diagram of a CAT bond. In case of the occurrence of an event (gray arrow), the SPV gives the principal back to the investors with the final coupon or the generous interest. In case of no event (black arrow), the SPV pays the ceding according to the terms of the reinsurance contract and sometimes pays nothing or partially the principal and interest to the investors.

There is a variety of trigger mechanisms to determine when the losses of a natural catastrophe should be covered by the CAT bond. These include the indemnity, the modeled loss, the industry index, the parametric index, the pure parametric and the hybrid trigger. Each of these mechanisms shows a range of levels of basis risks and transparency to investors.

The *Indemnity trigger* involves the actual loss of the ceding company. The ceding company receives reimbursement for its actual losses from the covered event, above

### 3 Catastrophe (CAT) Bonds

the predetermined level of losses. This trigger closely replicates the traditional reinsurance, but it is exposed to catastrophic and operational risk of the ceding company. In a *Modeled loss trigger* mechanism, after a catastrophe occurs the physical parameters of the catastrophe are used by a modelling firm to estimate the expected losses to the ceding company's portfolio. Instead of dealing with the company's actual claims, the transaction is based on the estimates of the model. If the modeled losses are above a specified threshold, the bond is triggered. With an *Industry index trigger*, the ceding company recovers a proportion of total industry losses in excess of a predetermined point to the extent of the remainder of the principal. The *Parametric index trigger* uses different weighted boxes to reflect the ceding company's exposure to events in different areas. The *Pure parametric index* payouts are triggered by the occurrence of a catastrophic event with certain defined physical parameters, e.g. wind speed and location of a hurricane or the magnitude or location of an earthquake. A *Hybrid trigger* uses more than one trigger type in a single transaction. The choice of a trigger for ILS involves a trade-off between moral hazard and basis risk Doherty [2000]. Pure indemnity triggers are subject to the highest degree of moral hazard, while the pure parametric triggers provide the lowest degree of moral hazard and highest basis risk. The modeled triggers is the intermediate case. Indices have also been developed that are hybrids of the four basic types or that incorporate more complicated functional forms than standard parametric indices. The use of an *industry index* loss trigger was by far the most popular mechanism for cat bonds in 2009, (Bloomberg, 2010).

The insurance linked securities market is still experimenting, but after the Katherine hurricane losses the CAT bond market has increased notably. The ILS market has expanded from covering catastrophes to other perils such as extreme mortality risk, automobile insurance, liability insurance, as well as life insurance securitizations, SwissRe [2006]

The pricing of CAT bonds reveals some similarities to the defaultable bonds, but CAT bonds offer higher returns because of the unfixed stochastic nature of the catastrophe process. The similarity between catastrophe und default in the log-normal context has been commented in Kau and Keenan [1996].

### 3.2 Calibrating a Mexican Parametric CAT Bond

As a result of its geographical location, Mexico finds itself under threat from a great variety of natural phenomena which can cause disasters, such as earthquakes, volcanic eruptions, hurricanes, forest fires, floods and aridity (dryness). In the event of a disaster, the effects on financial and natural resources are huge and volatile. Mexico's first priority is to transfer seismic risk, because although it is the less recurrent, it has the biggest impact on the population and the country. For example, an earthquake of magnitude 8.1 *Mw* Richter scale which hit Mexico in 1985, destroyed hundreds of buildings and caused thousands of deaths. Figure 3.2 depicts the number of earthquakes higher than 6.5 *Mw* occurred in Mexico during the years 1900-2003.

After the occurrence of a natural disaster, the reconstruction can be financed with

### 3 Catastrophe (CAT) Bonds

catastrophe bonds (CAT bonds) or reinsurance. For insurers, reinsurers and other corporations CAT bonds are hedging instruments that offer multi year protection without the credit risk present in reinsurance by providing full collateral for the risk limits offered through the transaction. For investors CAT bonds offer attractive returns and reduction of portfolio risk, since CAT bonds defaults are uncorrelated with defaults of other securities.

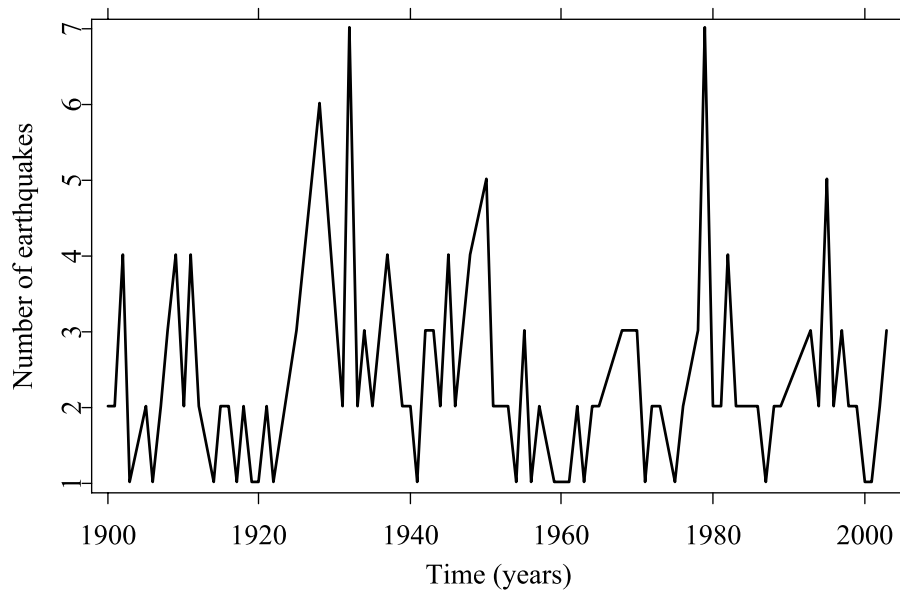


Figure 3.2: Number of Mexican earthquakes occurred during 1900-2003. In this figure, we plot the number of earthquakes higher than 6.5  $Mw$  occurred in Mexico during the years 1900-2003. Earthquakes less than 6.5  $Mw$  were not taken into account because of their high frequency and low loss impact.

Faced with the shortage of resources of the Mexico's Fund for Natural Disasters (FONDEN) and the high probability of earthquake occurrence, in May 2006 the Mexican government sponsored a pure parametric CAT bond against earthquake risk. The decision was taken because the instrument design protects and magnifies, with a degree of transparency, the resources of the trust and helps the government with emergency services and rebuilding after a big earthquake. For the transaction, FONDEN hired Swiss Re Capital Markets (SRCM), Swiss Reinsurance Company (SRC) and Deutsche Bank Securities.

The 160 million CAT bond is part of a 450 million insurance contract against earthquake risk covering equally three out of the nine zones (Zone 1, Zone 2 and Zone 5) in three years for a premium totaling 26 million, see Figure 3.3. The CAT bond payment would be triggered if there is an *event*, i.e. an earthquake higher than or equal to 8  $Mw$  hitting Zone 1 or Zone 2, or an earthquake higher than or equal to 7.5  $Mw$  hitting



### 3 Catastrophe (CAT) Bonds

Zone 5, thereby there is no justification of losses. The payment of losses is conditional upon confirmation by an event verification agent (Applied Insurance Research World-wide Corporation - AIR), which modelled the seismic risk. In case of event, SRC would pay the covered insured amount to the government, which would then stop paying premiums at that time and investors would sacrifice their full principal and coupons.

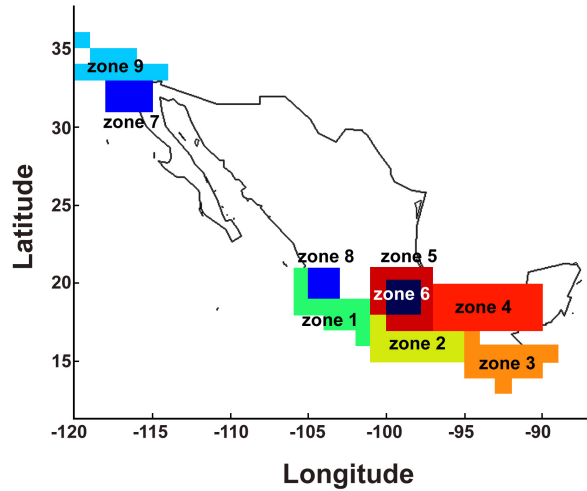


Figure 3.3: Map of seismic regions in Mexico. Source: de Hacienda y Crédito Público México [2004]

The cash flows diagram for the Mexican CAT bond is described in Figure 3.4, de Hacienda y Crédito Público México [2004]. SRCM designed a financial structure for transferring earthquake risk where FONDEN enters into a insurance contract with European Finance Reinsurance (EFR), an indirect, wholly-owned subsidiary of SRC which transfer 100% of the risk to SRC via the Reinsurance Agreement. SCR retained 290 million of the contract exposure and issued a 160 million CAT bond for three years through a special purpose vehicle, CAT-Mex Ltd class B insurer. From this part of the transaction, we infer the trigger intensity rate from the reinsurance market ( $\lambda_1$ ).

Simultaneously, CAT-Mex issues the bond with floating rate notes to capital markets investors to hedge its obligation to SRC under the financial contract and invests the proceeds in a total return swap within a collateral account Swiss Re Financial Products Corp., in exchange for quarterly LIBOR based payments. The larger tranche of the CAT bond (150 million class A) covers Zone 2 and has an expected annual loss of 0.96% and a spread over LIBOR of 235 basis points, while the smaller tranche (10 million class B notes) with coverage in Zone 1 and 5 has an expected annual loss of 0.93% and a spread over LIBOR of 230 basis points. A separate Event Payment Account was established with the Bank of New York providing FONDEN the ability to receive loss payments directly from CAT-MEX Ltd, subject to the terms and conditions of the insurance agreement. The trigger intensity rate from the capital market ( $\lambda_2$ ) is implied from this part of the transaction.

### 3 Catastrophe (CAT) Bonds

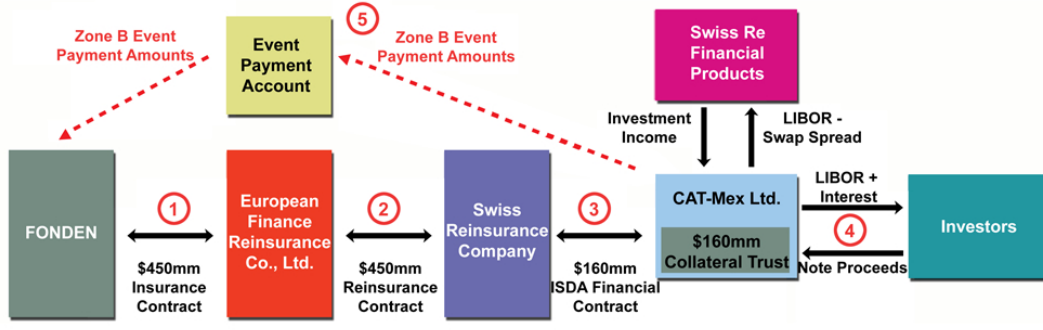


Figure 3.4: The cash flows diagram for the Mexican CAT bond. Source: de Hacienda y Crédito Público México [2004]

Assuming a perfect financial market, the calibration of the parametric CAT bond is based on the estimation of the intensity rate that describes the flow process of events, computed from the actuarial viewpoint  $\lambda_1$  (which balances the premiums and the expected losses), from the capital markets  $\lambda_2$  (where the cost of capital is the key) and from the historical data ( $\lambda_3$ ). First, let us describe the underlying process.

Let  $N_t, t \geq 0$  be the arrival process of earthquakes in the interval  $(0, t]$  defined as:

$$N_t = \sum_{n=1}^{\infty} \mathbf{1}(T_n < t) \quad (3.1)$$

where  $T_n$  describe the occurrence time of the  $n$ th earthquake. Since earthquakes can strike at any time during the year with the same probability, the traditional approach in seismology is to model earthquake recurrence as a random process  $X$  characterised by the loss of memory property  $P(X > x + y | X > y) = P(X > x)$ . Nevertheless it is possible to predict, on average, how many events will occur and how severe they will be. Therefore, the arrival process of earthquakes  $N_t$  can be modelled by an Homogeneous Poisson Process (HPP) with exponentially distributed intensity rate  $\lambda > 0$ . Hence, the probability of occurrence of an earthquake is:

$$P(\tau_i < t) = 1 - P(\tau_i \geq t) = 1 - \exp(-t\lambda) \quad (3.2)$$

with waiting times  $\tau_i = T_i - T_{i-1}$ . The occurrence of the first trigger event can be define as the stopping time  $\tau = \min \{t : N_t > 0\}$  occurred in insured zones with cdf  $F_{\tau(t)} = P(\tau < t) = P(N_t > 0) = 1 - \exp(-t\lambda)$  and  $f_{\tau(t)} = \lambda \exp(-t\lambda)$ .

#### 3.2.1 Calibration in the Reinsurance Market

Let  $J = 450 \cdot \mathbf{1}(\tau < 3)$  be a random variable with density function  $f_{\tau(t)}$  denoting the covered insured amount to the government in case of an event occurrence over a three year period  $T = 3$ . Let  $H$  be the total premium paid equal to 26 million. Suppose

### 3 Catastrophe (CAT) Bonds

a flat term structure of continuously compounded discount interest rates and a HPP with intensity rate  $\lambda_1$  describing the arrival process of earthquakes which trigger the bond. Under the risk neutral pricing measure, a compounded discount actuarially fair insurance price at time  $t = 0$  is defined as:

$$\begin{aligned} H &= E [J \exp(-\tau r_\tau)] = E [450 \cdot \mathbf{1}(\tau < 3) \exp(-\tau r_\tau)] \\ &= 450 \int_0^3 \exp(-tr_t) f_{\tau(t)} dt = 450 \int_0^3 \exp(-tr_t) \lambda_1 \exp(-t\lambda_1) dt \end{aligned} \quad (3.3)$$

i.e. the insurance premium is equal to the expected discounted value of losses caused from earthquakes. Substituting the values of  $H$  and assuming an annual continuously compounded discount interest rate  $r_t = \log(1.0541)$  constant and equal to the LIBOR in May 2006, we get:

$$26 = 450 \int_0^3 \exp \{-t \log(1.0541)\} \lambda_1 \exp(-t\lambda_1) dt \quad (3.4)$$

where  $1 - \exp(-t\lambda_1)$  is the event occurrence probability. The intensity rate of events from the reinsurance market  $\lambda_1$  is equal to 0.0214, meaning that the premium paid by the government to the insurance company considers a probability of event occurrence over three years equal to 0.0624 or the insurer expects 2.15 events in one hundred years.

#### 3.2.2 Calibration in the Capital Market

For computing the intensity rate of events in the capital market  $\lambda_2$ , we consider a coupon CAT paying bond with principal  $P = 160$  million at time to maturity  $T = 3$  years. The larger tranche of the CAT bond ( $P_2 = 150$  million class A) offers a spread  $z_2$  over LIBOR of 235 basis points covering Zone 2, while the smaller tranche ( $P_1 = 10$  million class B notes) gives  $z_1$  of 230 basis points over LIBOR covering Zone 1 and 5. We consider the annual discretely compounded discount interest rate  $r_t = 5.4139\%$  to be constant and equal to LIBOR in May 2006. We assume fixed quarterly coupons payments  $C_1$  for Zone 2 and  $C_2$  for Zone 1 and 5 during the bond's life in case of no event equal to:

$$\begin{aligned} C_1 &= \left( \frac{r_t + z_1}{4} \right) P_1 = \left( \frac{5.4139\% + 2.35\%}{4} \right) 150 = 2.9114 \\ C_2 &= \left( \frac{r_t + z_2}{4} \right) P_2 = \left( \frac{5.4139\% + 2.30\%}{4} \right) 10 = 0.1928 \end{aligned} \quad (3.5)$$

Since the coupons of each insured area are independent of each other, the total value of coupon payments is  $C = C_1 + C_2 = 3.1043$ . Assume that the arrival process of earthquakes which trigger the bond follows a HPP with intensity  $\lambda_2$ . Let  $G$  be a random variable defining the investors' payoff betting that a trigger event will not occur in Zone 2, or Zone 1 and Zone 5. Under the risk neutral pricing measure, the discretely discount

### 3 Catastrophe (CAT) Bonds

fair bond price at time  $t = 0$  is given by:

$$\begin{aligned}
 P &= E \left[ G \left( \frac{1}{1+r_\tau} \right)^\tau \right] \\
 &= E \left[ \sum_{t=1}^{12} C \cdot \mathbf{1}(\tau > \frac{t}{4}) \left( \frac{1}{1+r_t} \right)^{\frac{t}{4}} + P \cdot \mathbf{1}(\tau > 3) \left( \frac{1}{1+r_t} \right)^3 \right] \\
 &= \sum_{t=1}^{12} C \exp(-\lambda_2 \frac{t}{4}) \left( \frac{1}{1+r_t} \right)^{\frac{t}{4}} + P \exp(-3\lambda_2) \left( \frac{1}{1+r_t} \right)^3 \quad (3.6)
 \end{aligned}$$

Substituting the values of the principal  $P = 160$  million and the coupons  $C = 3.1043$  million in equation (3.6), it follows:

$$160 = \sum_{t=1}^{12} 3.1043 \left\{ \frac{\exp(-\lambda_2)}{1.0541} \right\}^{\frac{t}{4}} + \frac{160 \exp(-3\lambda_2)}{(1.0541)^3} \quad (3.7)$$

Solving equation (3.7), the intensity rate of events from the capital market  $\lambda_2$  is equal to 0.0241. In other words, the capital market estimates a probability of occurrence of an event equal to 0.0699, equivalently to 2.4 events in one hundred years.

#### 3.2.3 Calibration via Historical data

In addition to the estimation of the intensity rate of events for the reinsurance and the capital markets, the historical (physical) intensity rate  $\lambda_3$  that describes the flow process of events is calculated. The data was provided by the National Institute of Seismology in Mexico, de Geosifísica UNAM [2006]. It describes the time  $t$ , the depth  $d$ , the magnitude  $Mw$  and the epicenters of 192 earthquakes higher than 6.5  $Mw$  which occurred in the country from 1900 to 2003. Earthquakes less than 6.5  $Mw$  were not taken into account because of their high frequency and low loss impact. The data shows that 16% of the earthquakes greater than 6.5  $Mw$  had occurred in Zone 1, 22% in Zone 2, 9% in Zone 5 and 53% in other zones which are not insured.

Let  $Y_i$  be i.i.d. random variables, indicating the magnitude  $Mw$  of the  $i$ th earthquake at time  $t$ . The estimation of  $\lambda_3$  is based on the intensity model, which assumes the existence of i.i.d. random variables  $\varepsilon_i$  called *trigger events* that characterise earthquakes with magnitude  $Y_i$  higher than a defined threshold  $\bar{u}$  for a specific location, i.e.  $\varepsilon_i = \mathbf{1}(Y_i \geq \bar{u})$ . Consider the trigger event process  $B_t$  defined as:

$$B_t = \sum_{i=1}^{N_t} \varepsilon_i \quad (3.8)$$

where  $N_t$  is a HPP describing the arrival process of earthquakes with intensity  $\lambda > 0$ .  $B_t$  is a process which counts only earthquakes that trigger the CAT bond's payoff. However, the dataset contains only three such events, what leads to the calibration of the intensity of the process  $B_t$  be based on only two waiting times. Therefore in order

### 3 Catastrophe (CAT) Bonds

Year	Mw	Zone
1957	7.8	5
1985	8.1	1
1995	8.0	1

Table 3.1: Trigger events in earthquakes historical data.

100(1 - $\alpha$ )%	Confidence intervals
99%	0 - 0.0791
95%	0 - 0.0666
90%	0.0083 - 0.0594
50%	0.0183 - 0.0392

Table 3.2: Confidence Intervals for  $\lambda_3$  the intensity rate of events from the earthquake historical data.

to compute  $\lambda_3$ , consider the process  $B_t$  and define  $p$  as the probability of occurrence of a trigger event conditional on the occurrence of a trigger event conditional on the occurrence of an earthquake process with intensity  $\lambda$ . Then the probability of no event up to time  $t$  is equal to:

$$\begin{aligned}
P(B_t = 0) &= P(N_t = 0) + P(N_t = 1)(1 - p) + P(N_t = 2)(1 - p)^2 + \dots \\
&= \sum_{k=0}^{\infty} P(N_t = k)(1 - p)^k = \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} \exp(-t\lambda)(1 - p)^k \\
&= \sum_{k=0}^{\infty} \frac{\{\lambda(1 - p)t\}^k}{k!} \exp(-t\lambda) \exp\{-t\lambda(1 - p)\} \exp\{t\lambda(1 - p)\} \\
&= \exp(-tp\lambda) \\
&= \exp(-t\lambda_3)
\end{aligned} \tag{3.9}$$

by definition of the Poisson distribution and since  $\sum_{k=0}^{\infty} \frac{\{\lambda(1 - p)t\}^k}{k!} \exp\{-t\lambda(1 - p)\} = 1$ .

The calibration of the  $\lambda_3 = \lambda p$  is therefore decomposed into the calibration of the annual intensity of all earthquakes with magnitude higher than 6.5  $Mw$  ( $\lambda = 1.8504$ ) and the probability  $p$  of a trigger event (3 out of 192 earthquakes, see Table 3.1). Consequently  $\lambda_3 = 0.0289$ , meaning that approximately 2.89 events are expected to occur in the insured areas of the country within one hundred years.

Table 3.2 displays the confidence intervals for  $\lambda_3$ , which are very large since the value of  $\lambda_3$  depends on the time period of the historical data and the number of trigger events. Figure 3.5 shows the magnitude of earthquakes above 6.5  $Mw$  and trigger events occurred in Mexico during 1900 to 2003.

Table 3.3 summarises the values of the intensity rates  $\lambda$ 's and the probabilities of

### 3 Catastrophe (CAT) Bonds

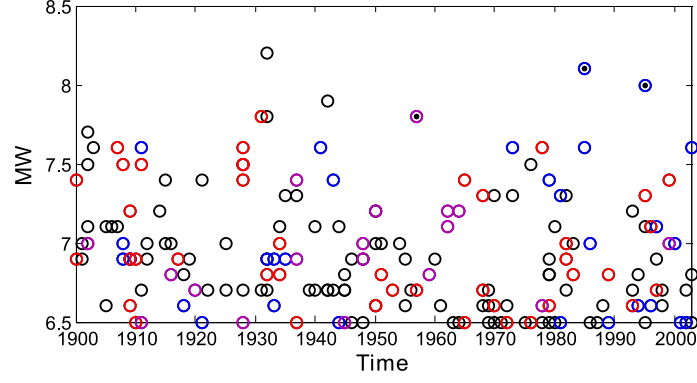


Figure 3.5: Magnitude of trigger events (filled circles), earthquakes occurred in Zone 1 (blue circles), earthquakes in Zone 2 (red circles), earthquakes in Zone 5 (magent circles) and earthquakes out of insured zones (blue circles)

	$\lambda_1$	$\lambda_2$	$\lambda_3$
Intensity	0.0214	0.0241	0.0289
Probability of event in 1 year	0.0212	0.0238	0.0284
Probability of event in 3 year	0.0624	0.0699	0.0830
No. expected events in 100 years	2.1482	2.4171	2.8912

Table 3.3: Calibration of intensity rates: the intensity rate  $\lambda_1$  from the reinsurance market, the intensity rate  $\lambda_2$  from the capital market and the historical intensity rate  $\lambda_3$ .

### 3 Catastrophe (CAT) Bonds

Rating	Data	1	2	3	4	5	6	7	8	9	10
AAA/Aaa	Mo	0.00	0.00	0.00	0.04	0.08	0.13	0.19	0.19	0.19	0.19
	S&P	0.00	0.00	0.09	0.19	0.29	0.43	0.50	0.62	0.66	0.70
AA/Aa	Mo	0.01	0.02	0.05	0.12	0.18	0.23	0.26	0.29	0.31	0.37
	S&P	0.01	0.05	0.10	0.20	0.32	0.43	0.56	0.68	0.78	0.89
A/A	Mo	0.20	0.10	0.25	0.38	0.51	0.64	0.76	0.89	1.01	1.09
	S&P	0.06	0.17	0.31	0.47	0.68	0.91	1.19	1.41	1.64	1.90
BBB/Baa	Mo	0.21	0.57	1.00	1.53	2.06	2.57	3.70	3.51	3.92	4.37
	S&P	0.24	0.71	1.23	1.92	2.61	3.28	3.82	4.38	4.89	5.42
BB/Ba	Mo	1.27	3.50	6.20	8.89	11.26	13.37	15.26	16.95	18.44	19.83
	S&P	1.07	3.14	5.61	7.97	10.10	12.12	13.73	15.15	16.47	17.49
B/B	Mo	5.26	11.44	17.31	22.41	27.26	31.59	35.50	38.80	41.59	43.80
	S&P	4.99	10.92	15.90	19.76	22.55	24.72	26.54	28.00	29.20	30.42
CCC/Caa	Mo	17.14	28.13	37.62	45.34	50.89	55.00	57.76	60.65	64.79	71.27
	S&P	26.29	34.73	39.96	43.19	46.22	47.49	48.61	49.23	50.95	51.83

Table 3.4: Cumulative Default Rate comparison from Moody's (Mo) and Standard and Poor's S&P (in % for up to 10 years).

occurrence of a trigger event in one and three years. Whereas the reinsurance market expects approximately 2.15 events to occur in one hundred years, the capital market anticipates 2.42 events and the historical data predicts 2.89 events. In other words, the CAT bond has a 6.99% chance of default (or event) within 3 years, which rates in the range of "BB" by Standard and Poor (2007) or "Ba" by Moody's Investors Services (2007), see Table 3.4.

Our estimation of intensity rates show that by transferring the seismic risk with a combination of reinsurance and CAT bond (Market insurance) is optimal in the sense that it provides coverage for a lower cost and lower exposure at default than reinsurance itself. First, under the risk neutral world, the implied risk trigger rates computed from the reinsurance and capital markets are reliable since their values lie inside the confidence intervals of the physical trigger rate  $\lambda_3$ , meaning that the direct access of the CAT bond into Capital Markets expand the risk bearing capacity beyond the limited capital held by reinsurers. The government paid a 26 million premium that is equivalent to a 0.75 times the 34.605 million real actuarially fair one ( $\lambda_3$  in equation (3.4)). This is contrary to the theory predictions of charging higher premiums by reinsurers after a catastrophe has occurred. The difference in reinsurance premiums is explained by the market price of risk, which in this case is negative. Since  $\lambda_3$  is only 50% confident, no further analysis about market price of risk will be done in our analysis.

Second, the difference between  $\lambda_1$  and  $\lambda_2$  can be explained by the absence of the public and liquid market of earthquake risk in the reinsurance market, since just limited information is available. This cause the pricing in the reinsurance market to be less transparent than pricing in the capital market. Another argument to this difference is that contracts in the capital market are more expensive than contracts in the reinsurance market, e.g. when one considers the default risk or the cost of risk capital (the required return necessary to make a capital budgeting project). The cost of risk capital in the capital market is usually higher than that in the reinsurance market (note that this is a

### 3 Catastrophe (CAT) Bonds

key for the estimation of  $\lambda_2$ ).

Since the insured loss faced by the government is independent of the state of insolvency of the reinsurer, our results indicate that the default probability of the reinsurer in this transaction over the next three years can be approximately equal to the price discount that the government gets for transferring the earthquake risk from the reinsurance market to the capital market ( $\approx 10.7\%$ ). In other words, the reinsurer default probability is implied by the relative difference of the premiums estimated in the reinsurance and the capital markets. For the case of 100% reinsurance (market insurance), the exposure at default is approximately equal to 10.7% of 450 million coverage (10.7% of 290 million) and the expected loss equals to  $10.7\% \times 450 \times (1 - \text{recovery rate})$  ( $10.7\% \times 290 \times (1 - \text{recovery rate})$ ). Since recovery rates are not very accurately estimated, we neglect the computation of the expected loss. Observe also that in the market insurance, the CAT bond does not present credit risk as the proceeds of the bond are held in a SPV, a transaction off the insurer's balance sheet. Therefore, the total paid premium of 26 million consists of 18.799 million ( $\int_0^3 290\lambda_2 \exp\{-t(r_t + \lambda_2)\} dt$ ) market insurance layer and 7.221 million from transaction costs or for coupon payments.

Finally, the mix of reinsurance and CAT bond not only eliminates restrictions in the budget when considering interest payments, but also protects and magnifies the resources of the trust. However, this financial strategy does not eliminate completely the costs imposed by market imperfections.

### 3.3 Pricing modelled-index CAT bonds for Mexican earthquakes

In this section, under the assumptions of non-arbitrage and continuous trading, we examine the pricing of a CAT bond for earthquakes occurred in whole Mexico with a modelled-index loss trigger mechanism. In essence, this hybrid trigger combines modelled loss and index trigger types, trying to reduce moral and basis risk borne by the sponsor, while remaining a non-indemnity trigger mechanism. This time the payout of the bond will be based on the historical and estimated losses, which are affected by several variables, for example: the magnitude  $Mw$  and the depth  $DE$  of the earthquake or its impact on main cities  $IMP(0, 1)$ , among others. Due to the identical geodetic conditions outside or inside the insured zones, it makes sense to use also the frequency of earthquakes occurred out of the insured areas.

We applied the pricing CAT bond methodology of Baryshnikov et al. [2001] and Burnecki and Kukla [2003] to the loss data that we have built for the Mexican earthquake data obtained from the National Institute of Seismology in Mexico (SSN). The pricing methodology is based on the characteristics of the dominant underlying processes, therefore we fit the distribution function of the incurred losses  $F(x)$  and the process  $N_t$  governing the flow of earthquakes.



### 3 Catastrophe (CAT) Bonds

Descriptive	$t$	$DE$	$Mw$	$X$ (\$ million)
Minimum	1900	0.00	6.50	0.00
Maximum	2003	200.00	8.20	1443.69
Mean	1951	39.54	6.93	10.73
Median	1950	33.00	6.90	0.00
Sdt. Error	-	39.66	0.37	105.16
25% Quantile	1928	12.00	6.60	0.00
75% Quantile	1979	53.00	7.10	0.00
Skewness	-	1.58	0.92	13.19
Kurtosis	-	5.63	3.25	179.52
Nr. obs.	192	192.00	192.00	192.00
Distinct obs.	82	54.00	18.00	24.00

Table 3.5: Descriptive statistics for the variables time  $t$ , depth  $DE$ , magnitude  $Mw$  and loss  $X$  of the loss historical data

#### 3.3.1 Severity of Mexican earthquakes

The loss data contains information of 192 earthquakes higher than 6.5  $Mw$  occurred in Mexico during 1900 to 2003, 24 of them with financial adjusted losses. There are two outliers with losses: the 8.1  $Mw$  earthquake in 1985 and the 7.4  $Mw$  earthquake in 1999. The earthquake in 1932 had the highest magnitude in the historical data (8.2  $Mw$ ), but its losses are not big enough compared to the other earthquakes.

The historical losses were adjusted to the population growth, the inflation rate, the exchange rate (peso/dollar) and converted to USD in 1990. In order to do that, we used the annual population per Mexican Federation (1900-2003) provided by the National Institute of Geographical and Information Statistics in Mexico (INEGI), the annual Consumer Price Index (1860-2003) and the Average Parity Dollar-Peso (1821-1997), both provided by the U.S. Department of Labour. The descriptive statistics for the variables time  $t$ , depth  $DE$ , magnitude  $Mw$  and adjusted loss  $X$  of the earthquake data are given in Table 3.5.

We observe that the log of historical adjusted losses,  $\log(X)$ , are directly proportional to the time  $t$  and the magnitude  $Mw$ , and inversely proportional to the depth  $DE$  of the earthquake. When the outliers are excluded, the adjusted losses in log-terms are inversely proportional to  $t$ ,  $Mw$  and  $DE$ . Given these facts, we model the losses by means of linear and robust regression methods. However, the robust regression shows similar results as in the case of the linear regression without outliers. Under the selection criterion of the highest coefficient of determination  $R^2$ , the best linear regression loss model with and without outliers are of the next form:

$$\log(X) = \beta_0 + \beta_1 Mw + \beta_2 DE + \beta_3 IMP(0,1) + \beta_4 \log(Mw) \cdot DE \quad (3.10)$$

where  $IMP(0,1)$  indicates the impact of the earthquake on Mexico City. The coefficients of the linear regression loss models, its corresponding coefficients of determination  $R^2$

### 3 Catastrophe (CAT) Bonds

<i>No – Outliers</i>	$\beta_0$	$\beta_1$	$\beta_2$	$\beta_3$	$\beta_4$	$R^2$	$SE$
	-27.99	2.10	4.44	-0.15	-1.11	0.22	2.86
EQ-1985	-7.38	0.97	1.51	-0.19	-0.52	0.15	2.83
EQ-1985,1999	1.30	0.40	0.23	0.18	-0.23	0.129	2.83

Table 3.6: The coefficients of the linear regression loss models and its corresponding coefficients of determination  $R^2$  and standard errors  $SE$  for the modelled loss data with, without the earthquake in 1985 (EQ-1985) and without the earthquake in 1999 (EQ-1985,1999).

and standard errors are shown in Table 3.6.

Since 23 out of 192 observations have information about earthquake losses, we treat the missing loss data with the Expectation - Maximum algorithm (EM) with linear regression, Howell [1998]. Figure 3.6 shows the historical and estimated losses of earthquakes.

In order to find an accurate loss distribution that fits the loss data, we compare the shapes of the theoretical mean excess function  $e_x$ :

$$e_x = E[X - x | X > x] = \frac{\int_x^\infty \{1 - F_u\} du}{1 - F_x} \quad (3.11)$$

with the empirical

$$\hat{e}_{n(x)} = \frac{\sum_{x_i > x} x_i}{\#i : x_i > x} - x$$

Figure 3.7 shows the log mean excess function as a function of log loss. The left panel shows an increasing pattern of the  $\log \{\hat{e}_{n(x)}\}$  for the log loss data, pointing out that the distribution of losses have heavy tails. In this case, Hogg and Klugman [1984], Klugman et al. [2008] and Grossi and Kunreuther [2005] indicate that the Log-normal, the Burr or the Pareto distribution are candidates to be the analytical distribution of the loss data. Alternatively, one can use the g-and-h distribution as a parametric model for model the severity of losses (Dutta and Perry [2007]), but over large ranges the g-and-h distribution behaves approximately like an exact Pareto distribution, see Degen and Embrechts [2008]. The right panel of Figure 3.7 shows a decreasing pattern of the  $\log \{\hat{e}_{n(x)}\}$  for the log loss data without the earthquake in 1985, indicating that Gamma, Weibull or Pareto can model adequately.

Next, we apply the *edf* statistics: Kolmogorov Smirnov (D test), the Kuiper statistic (V test), the Cramér-von Mises ( $W^2$  test) and the Anderson Darling ( $A^2$  test) non-parametric tests, to check goodness of fit.

Under the null hypothesis  $\{H_0 : F_{n(x)} = F_{(x,\theta)}\}$ , where  $F_{n(x)} = \frac{1}{n} \# \{i : x_i \leq x\}$  and  $\theta$  be a vector of unknown parameters.  $\theta$  can be estimated by simply finding a  $\hat{\theta}^*$  that min-

### 3 Catastrophe (CAT) Bonds

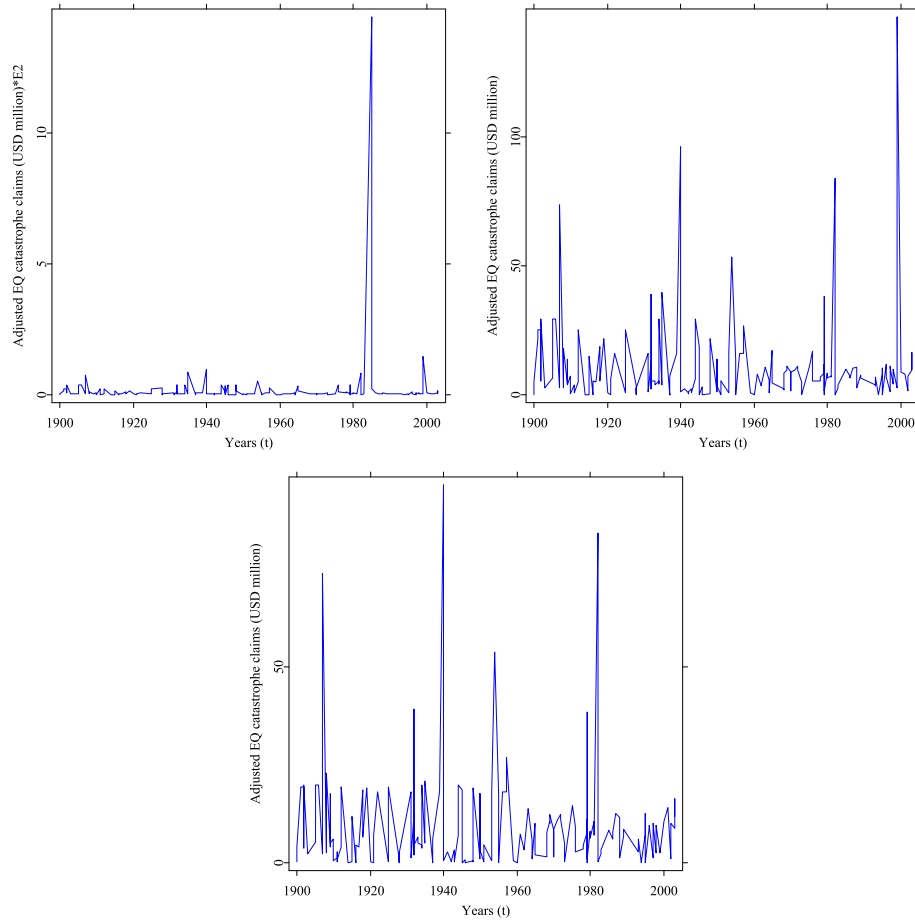


Figure 3.6: Historical and modelled losses of Mexican earthquakes during 1900-2003 (upper left panel), without the earthquake in 1985 (upper right panel), without the earthquakes in 1985 and 1999 (lower panel)

### 3 Catastrophe (CAT) Bonds

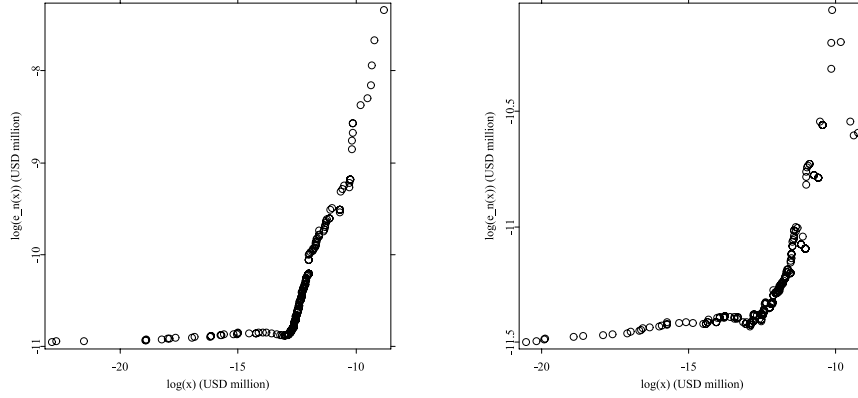


Figure 3.7: The log of the empirical mean excess function  $\log \left\{ \hat{e}_{n(x)} \right\}$  for the modelled loss data with (left panel) and without the outlier of the earthquake in 1985 (right panel).

imizes a particular *edf* statistic. D'Agostino and Stephens [1986] state that when the fitted distribution  $F_x$  diverge from the true distribution in the tails, the  $A^2$  test is the most potent statistic since it gives more weights in the tails than the above statistics and it returns lower *edf* test statistics values than the maximum likelihood algorithm.

The estimated parameters of the modelled loss data via  $A^2$  statistic minimization, the corresponding *edf* test statistics and their *p*-values based on 1000 simulated samples of the modelled loss data with and without the outlier of the 1985 earthquake are shown in Table 3.7. Due to the quality of the loss data, the tests reject the fit for all the distributions. Although the  $A^2$  statistic does not pass any distribution for the loss model (3.10) in case of no outliers, it passes the Burr distribution for other loss models. The Pareto distribution passes all the test for other loss models. When the data does not take into account the 1985 earthquakes data, the Exponential, Pareto and Gamma distribution pass the Cramér-von M. statistic for the loss model (3.10), while the Weibull distribution passes all the test statistics for other loss models. All the remaining distributions give worse fits.

For a fixed amount deductible of  $x$ , the limited expected value function characterises the expected amount per loss retained by the insured in a policy, Hogg and Klugman [1984] and Klugman et al. [2008]:

$$l_x = E \{ \min(X, x) \} = \int_0^x y dF_y + x (1 - F_x), x > 0 \quad (3.12)$$

where  $X$  is the loss amount random variable with cdf  $F(x)$ . The empirical estimate is calculated as  $\hat{l}_{n(x)} = \frac{1}{n} \left( \sum_{x_j < x} x_j + \sum_{x_j \geq x} x \right)$ .  $l_x$  not only fits but also emphasises how different parts of the loss distribution function contribute to the premium. Figure 3.8

### 3 Catastrophe (CAT) Bonds

Distrib.	Log-normal	Pareto	Burr	Exponential	Gamma	Weibull
Parameter	$\mu = 1.456$ $\sigma = 1.677$	$\alpha = 2.199$ $\lambda = 12.53$	$\alpha = 3.354$ $\lambda = 17.33$ $\tau = 0.895$	$\beta = 0.132$	$\alpha = 0.145$ $\beta = -0.0$	$\beta = .214$ $\tau = .747$
Kolmogorov S.	0.185	0.142	0.150	0.149	0.299	0.157
(D test)	(< 0.005)	(< 0.005)	(< 0.005)	(< 0.005)	(< 0.005)	(< 0.005)
Kuiper	0.308	0.265	0.278	0.245	0.570	0.298
(V test)	(< 0.005)	(< 0.005)	(< 0.005)	(< 0.005)	(< 0.005)	(< 0.005)
Cramér-von M.	1.447	0.879	0.987	0.911	6.932	1.16
( $W^2$ test)	(< 0.005)	(< 0.005)	(< 0.005)	(< 0.005)	(< 0.005)	(< 0.005)
Anderson D.	10.490	6.131	6.018	10.519	35.428	6.352
( $A^2$ test)	(< 0.005)	(< 0.005)	(< 0.005)	(< 0.005)	(< 0.005)	(< 0.005)
EQ-85						
Parameter	$\mu = 1.493$ $\sigma = 1.751$	$\alpha = 2.632$ $\lambda = 17.17$	$\alpha = 1.8e7$ $\lambda = 9.5e7$ $\tau = 0.770$	$\beta = 0.120$	$\alpha = 0.666$ $\beta = .070$	$\beta = 0.194$ $\tau = .770$
Kolmogorov S.	0.116	0.077	0.070	0.081	0.070	0.070
(D test)	(< 0.005)	(< 0.005)	(0.001)	(0.084)	(< 0.005)	(0.008)
Kuiper	0.215	0.133	0.126	0.138	0.121	0.126
(V test)	(< 0.005)	(0.006)	(< 0.005)	(0.008)	(< 0.005)	(< 0.005)
Cramér-von M.	0.702	0.168	0.166	0.202	0.147	0.166
( $W^2$ test)	(< 0.005)	(0.012)	(< 0.005)	(0.152)	(0.006)	(< 0.005)
Anderson D.	6.750	3.022	1.617	4.732	1.284	1.617
( $A^2$ test)	(< 0.005)	(< 0.005)	(< 0.005)	(< 0.005)	(< 0.005)	(< 0.005)

Table 3.7: Parameter estimates by  $A^2$  minimization procedure and test statistics for the modelled loss data with and without the 1985 earthquake outlier (EQ-85). In parenthesis, the related  $p$ -values based on 1000 simulations.

### 3 Catastrophe (CAT) Bonds

presents log-log plots of  $l_x$  and  $\hat{l}_{n(x)}$  for the analysed data set with (left panel) and without the 1985 earthquake data (right panel). The closer they are, the better they fit and the closer the mean values of both distributions are. The situation depicted in Figure 3.8 motivates to base the further analysis on the Burr, Pareto, Gamma and Weibull distributions.

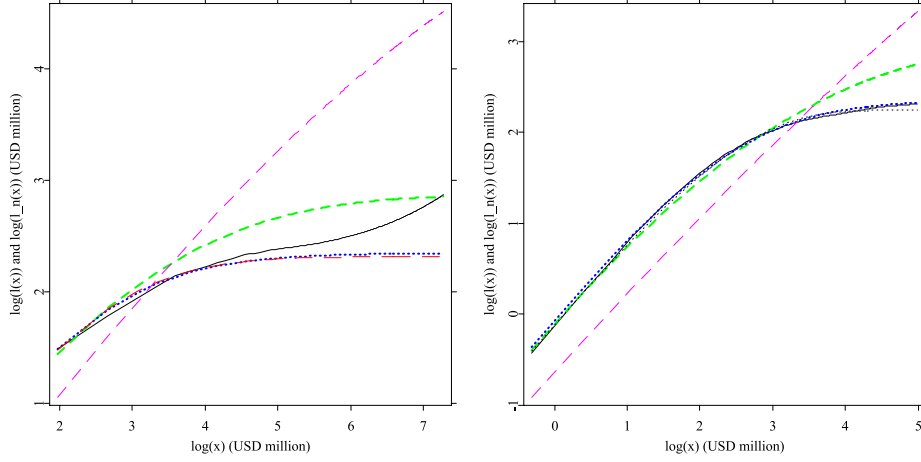


Figure 3.8: The log of the empirical limited expected value function  $\log \left\{ \hat{l}_{n(x)} \right\}$  (black solid line) and the log of the theoretical limited expected value function  $\log(l_x)$  for the log-normal (green dashed line), Pareto (blue dashed line), Burr (red dashed line), Weibull (magenta dashed line) and Gamma (black dashed line) distributions for the modelled loss data with (left panel) and without the outlier of the 1985 earthquake (right panel).

#### 3.3.2 Frequency of Mexican earthquakes

In this section we focus on the efficient simulation of the arrival point process of earthquakes  $N_t$ . We first look for the appropriate shape of the approximating distribution. We achieve that by examining the empirical mean excess function  $\hat{e}_{n(t)}$  for the waiting times of the earthquake data. The left panel of Figure 3.9 shows the log of the empirical mean excess as a function of log time. It shows increasing starting and a decreasing ending behaviour, implying that the Exponential, Gamma, Pareto and Log-normal distribution are possible candidates to fit the arrival process of earthquakes. The right panel of Figure 3.9 shows the log of the theoretical mean excess function  $e_t$  increasing with  $\log(t)$ , meaning that the tails of the analytical distributions candidates are different from the tails of the empirical distribution.

We estimate the parameters of the candidate analytical distributions via the  $A^2$  minimisation procedure and test the Goodness of fit for the arrival process of earthquakes. The estimated parameters and their corresponding  $p$ -values based on 1000 simulations

### 3 Catastrophe (CAT) Bonds

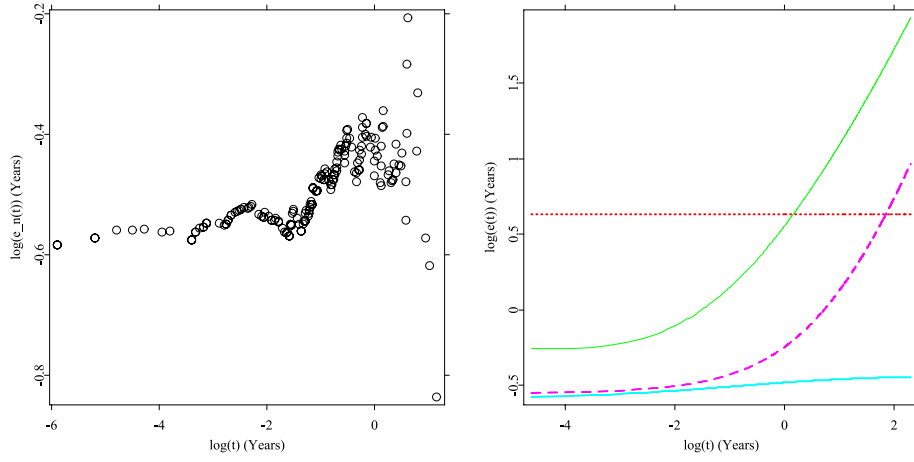


Figure 3.9: The log of the empirical mean excess function  $\log \{ \hat{e}_n(t) \}$  for the waiting times of the earthquakes data (left panel) and the log of the theoretical mean excess function  $\log (e_t)$  for the fitted log-normal (green solid line), exponential (red dotted line), Pareto (magenta dashed line) and Gamma (cyan solid line) distributions (right panel)

are illustrated in Table 3.8. Observe that the Exponential, Pareto and Gamma distributions pass all the tests at a very high level.

When the arrival process of earthquakes is modelled with a HPP, the intensity rate  $\lambda = 1.8504$  and it is independent of time and the estimation of the annual intensity rate of earthquakes higher than 6.5  $M_w$  is equal to 1.8504. Comparing this annual intensity rate with the one obtained by the fitted Exponential distribution ( $\lambda = 1.88$ ) indicates that the earthquakes arrival process can be correctly modelled with a HPP.

We also model the earthquake arrival process with a Non-homogeneous Poisson Process (NHPP). This time, the expected value is equal to  $E(N_t) = \int_0^t \lambda_s ds$ , where the intensity rate  $\lambda_s$  is dependent on time. We tested different polynomial functions to model the time varying  $\lambda_s$ , however the constant intensity rate  $\lambda_s = 1.8167$  with a coefficient of determination  $R^2 = 0.99$  and standard error  $SE = 2.33$  turned out to be the best fit. This result shows that the HPP suffices in modelling the arrival process of earthquakes. Figure 3.10 depicts the accumulated number of earthquakes and the mean value functions  $E[N_t]$  of the HPP with intensity rates  $\lambda = 1.8504$  and  $\lambda_s = 1.8167$ .

#### 3.3.3 Pricing modelled-Index CAT bonds

We price an index CAT bond by means of the compound doubly stochastic Poisson pricing methodology, Baryshnikov et al. [2001]. The CAT bond pricing relies on the statistical characteristics of the dominant underlying processes. The stochastic assumptions are:

### 3 Catastrophe (CAT) Bonds

Distrib.	Log-normal	Exponential	Pareto	Gamma
Parameter	$\mu = -1.158$ $\sigma = 1.345$	$\beta = 1.880$	$\alpha = 5.875$ $\lambda = 2.806$	$\alpha = 0.858$ $\beta = 1.546$
Kolmogorov S.	0.072	0.045	0.035	0.037
(D tests)	(0.005)	(0.538)	(0.752)	(0.626)
Kuiper	0.132	0.078	0.067	0.064
(V test)	(< 0.005)	(0.619)	(0.719)	(0.739)
Cramér-von M	0.212	0.062	0.031	0.030
( $W^2$ test)	(< 0.005)	(0.451)	(0.742)	(0.730)
Anderson D.	2.227	0.653	0.287	0.190
( $A^2$ test)	(< 0.005)	(0.253)	(0.631)	(0.880)

Table 3.8: Parameter estimates by  $A^2$  minimization procedure and test statistics for the earthquake data. In parenthesis, the related  $p$ -values based on 1000 simulations.

1. There is a doubly stochastic Poisson process  $N_s$  (a Poisson process conditional on a stochastic intensity process  $\lambda_s$ ) with  $s \in [0, T]$ , describing the flow of a particular catastrophic natural event in a specified region.
2. The financial losses  $\{X_k\}_{k=1}^{\infty}$  caused by these catastrophic events are i.i.d random variables with cdf  $F_X$ .
3. A continuous and predictable aggregate loss process is defined as:

$$L_t = \sum_{i=1}^{N_t} X_i \quad (3.13)$$

where  $N_s$  and  $X_k$  are assumed to be independent processes.

4. A continuously compounded discount interest rate  $r$  describing the value at time  $s$  of 1 USD paid at time  $t > s$  by  $\exp\{-R(s, t)\} = \exp\left\{-\int_s^t r(\xi)d\xi\right\}$
5. A threshold time event  $\tau = \inf\{t : L_t \geq D\}$  that is the moment when the aggregate loss  $L_t$  exceeds the threshold level  $D$ . Baryshnikov et al. [2001] defines the threshold time as a doubly stochastic Poisson process  $M_t = \mathbf{1}(L_t > D)$ , with a stochastic intensity depending on the index position:

$$\Lambda_s = \lambda_s \left\{1 - F_{(D-L_s)}\right\} \mathbf{1}(L_s < D) \quad (3.14)$$

Under these assumptions, assume a zero coupon CAT bond (ZCCB) that pays a principal amount  $P$  at time to maturity  $T$ , conditional on the threshold time  $\tau > T$ . Let  $\mathcal{F}_t$  be an increasing filtration with time  $t \in [0, T]$  and let  $P$  be a predictable process  $P_s = E(P|\mathcal{F}_s)$ , i.e. the payment at maturity is independent of the occurrence and timing of the threshold  $D$ . The principal  $P$  is fully lost in case of occurrence of a trigger event.



### 3 Catastrophe (CAT) Bonds

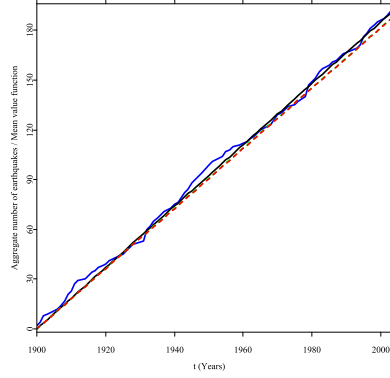


Figure 3.10: The accumulated number of earthquakes (solid blue line) and mean value functions  $E[N_t]$  of the Homogeneous Poisson Process (HPP) with the constant intensity  $\lambda = 1.8504$  (solid black line) and the time dependent intensity  $\lambda_s = 1.8167$  (dashed red line).

The non arbitrage price of the ZCCB ( $V_t^1$ ) associated with the threshold  $D$ , the earthquake flow process  $N_s$  with intensity rate  $\lambda_s$ , a loss distribution function  $F_x$  is given by:

$$\begin{aligned} V_t^1 &= E[P \exp\{-R(t, T)\} (1 - M_T) | \mathcal{F}_t] \\ &= E\left[P \exp\{-R(t, T)\} \left\{1 - \int_t^T \lambda_s (1 - F_{D-L_s}) \mathbf{1}(L_s < D) ds\right\} | \mathcal{F}_t\right] \end{aligned} \quad (3.15)$$

i.e. the ZCCB price is equal to the expected discounted value of the principal  $P$  contingent on the threshold time  $\tau > T$ . The doubly stochastic Poisson process incorporates the rate and impact of the catastrophe process.

Under the same assumptions, the non arbitrage price of a coupon CAT bond (CCB)  $V_t^2$  that pays the principal  $P$  at time to maturity  $T$  and gives coupon  $C_s$  with a fixed spread  $z$  over LIBOR until the threshold time  $\tau$  is given by:

$$\begin{aligned} V_t^2 &= E\left[P \exp\{-R(t, T)\} (1 - M_T) + \int_t^T \exp\{-R(t, s)\} C_s (1 - M_s) ds | \mathcal{F}_t\right] \\ &= E\left[P \exp\{-R(t, T)\} \right. \\ &\quad + \int_t^T \exp\{-R(t, s)\} \left(C_s \left\{1 - \int_t^s \lambda_{\xi} (1 - F_{D-L_{\xi}}) \mathbf{1}(L_{\xi} < D) d\xi\right\} \right. \\ &\quad \left. \left. - P \exp\{-R(s, T)\} \lambda_s (1 - F_{D-L_s}) \mathbf{1}(L_s < D)\right\} ds | \mathcal{F}_t\right] \end{aligned} \quad (3.16)$$

### 3 Catastrophe (CAT) Bonds

Quantile	3 years accumulated loss
10%	18.447
20%	23.329
30%	32.892
40%	44.000
50%	61.691
60%	80.458
<b>70%</b>	<b>109.110</b>
<b>80%</b>	<b>119.860</b>
90%	142.720
100%	1577.600

Table 3.9: Quantiles of 3 years accumulated modelled losses

Following this pricing methodology, we obtain the values of a (Z)CCB with principal  $P = 160$  million at  $t = 0$  with respect to the threshold level  $D$  and expiration time  $T \in [0, 3]$  years. First, we define the threshold  $D \in [100, 135]$  million, which corresponds to the 0.7 and 0.8-quantiles of the three yearly accumulated modelled losses (three payoffs expected approximately to occur in one hundred years), see Table 3.9. Second, we consider a continuously compounded discount interest rate  $r = \log(1.054139)$  to be constant and equal to the LIBOR in May 2006. We assume the Burr and Pareto distributions as the loss distributions for the modelled loss data from (3.10), and the Gamma, Pareto and Weibull distributions for the modelled loss data without the 1985 earthquake data. An HPP with constant intensity rate  $\lambda_s = \lambda = 1.8504$  describes the arrival process of earthquakes.

After applying 1000 Monte Carlo simulations, the results show that the price of the ZCCB decreases as  $T$  increases because the occurrence probability of the trigger event increases, and it increases as the  $D$  increases since one expects a trigger event with low probability. When  $D = 135$  million and  $T = 1$  year, the ZCCB price  $160e^{-\log(1.054139)} \approx 151.78$  million is equal to the case when the threshold time  $\tau = \inf \{t : L_t > D\}$  is greater than the maturity  $T$  with probability one. Although the ZCCB prices are pretty similar for the modelled loss data, they are higher and less volatile in the Pareto distribution (Std. deviation = 10.06) than under the Burr distribution (Std. deviation = 10.8). Without the outlier in the modelled loss data, the Gamma distribution leads to higher prices than the Weibull and Pareto distributions with lower standard deviations (8.83, 10.44 and 9.05 respectively).

For a CCB, we consider the same assumptions of the ZCCB and a spread rate  $z$  equal to 235 basis points over LIBOR. The bond has quarterly annual coupons  $C_{\frac{t}{4}} = \left( \frac{\text{LIBOR} + 235bp}{4} \right) 160 = 3.1055$  million. After 1000 Monte Carlo simulations, the price of the CCB at  $t = 0$  with respect to the threshold level  $D$  and expiration time  $T$  is computed for the Burr, Pareto, Gamma and Weibull distribution of the modelled loss data with and without the outlier of the earthquake in 1985. The increase in expiration time  $T$

### 3 Catastrophe (CAT) Bonds

raises the probability of a trigger event, causing lower price, but simultaneously more coupon payments are expected to be received, causing higher price. Therefore, CCB price will decline only when the former effect dominates the latter.

Concerning to the loss distributions functions for the modelled loss data, the Pareto distribution leads to higher CCB prices and lower standard deviation than the Burr distribution (Std. = 8.15 and 8.31 respectively). While for the modelled loss data without the outlier of the earthquake in 1985, the Gamma distribution offered higher prices and lower standard deviation than the Weibull and Pareto distributions (Std. deviations = 6.39, 8.62 and 7.24. respectively). Figure 3.11 shows the CCB price computed under the Burr distribution. Table 3.10 shows the minimum and maximum of the differences in (Z)CCB prices (in % of principal) for the analytical loss ditribution candidates of the modelled loss data with and without the 1985 earthquake data. Note that the CCB prices are higher than the ZCCB prices.

Burr - CAT Bond Prices

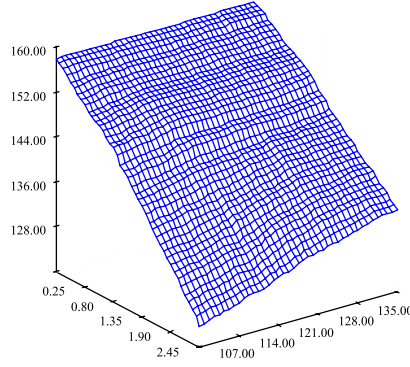


Figure 3.11: Coupon CAT bond prices (vertical axis) with respect to the threshold level (horizontal right axis) and expiration time (horizontal left axis) under the Burr distribution and a Homogeneous Poisson Process.

In order to verify the robustness of the loss models with respect to the (Z)CCB prices, we compare the bond prices calculated from different loss models with the bond prices obtained from simulations of the pricing algorithm. Let  $\hat{P}^*$  be the reference price or the (Z)CCB prices of the best loss model and let  $\hat{P}_i$  ( $i = 1 \dots, m > 0$ ) be the (Z)CCB prices from the  $i$ th loss model.  $\hat{P}^*$  and  $\hat{P}_i$  ( $\hat{P}^* \neq \hat{P}_i$ ) are generated using the same seed of the pseudorandom number generator in 1000 Monte Carlo simulations. Furthermore, let  $\hat{P}_j$  ( $j = 1 \dots n, n > 0$ ) be the  $j$ th simulated (Z)CCB price generated (with different seed) from 1000 trajectories of the (Z)CCB prices bond with the best loss model.

### 3 Catastrophe (CAT) Bonds

<i>No – Outliers</i>	ZCCB/CCB	Distribution	Min. (% Principal)	Max. (% Principal)
-	ZCCB	Burr-Pareto	-2.640	0.614
EQ-85	ZCCB	Gamma-Pareto	0.195	4.804
EQ-85	ZCCB	Pareto-Weibull	-4.173	-0.193
EQ-85	ZCCB	Gamma-Weibull	-0.524	1.636
-	CCB	Burr-Pareto	-1.552	0.809
EQ-85	CCB	Gamma-Pareto	0.295	6.040
EQ-85	CCB	Pareto-Weibull	-3.944	-0.295
EQ-85	CCB	Gamma-Weibull	-0.273	3.105
-	ZCB-CCB	Burr	-6.228	-0.178
-	ZCCB-CCB	Pareto	-5.738	-0.375
EQ-85	ZCCB-CCB	Gamma	-7.124	-0.475
EQ-85	ZCCB-CCB	Pareto	-5.250	-0.376
EQ-85	ZCCB-CCB	Weibull	-5.290	-0.475

Table 3.10: Minimum and maximum of the differences in the (Zero) Coupon CAT bond prices ((Z)CCB) (in % of principal), for the Burr-Pareto distributions of the modelled loss data and the Gamma-Pareto-Weibull distributions of the modelled loss data without the outlier of the earthquake in 1985 (EQ-85).

We compute the mean of the absolute differences (MAD) as:

$$\sum_{i=1}^m \frac{\hat{P}_i - \hat{P}^*}{m} \simeq \sum_{j=1}^n \frac{\hat{P}_j - \hat{P}^*}{n}, m > 0, n > 0 \quad (3.17)$$

meaning that if the MAD's are similar then the type of the model has no influence on the prices of the (Z)CCB prices. In terms of relative differences, if the means of the absolute values of the relative differences (MAVRD) are similar then the loss model has no impact on the (Z)CCB prices:

$$\sum_{i=1}^m \frac{1}{m} \left| \frac{\hat{P}_i - \hat{P}^*}{\hat{P}^*} \right| \simeq \sum_{j=1}^n \frac{1}{n} \left| \frac{\hat{P}_j - \hat{P}^*}{\hat{P}^*} \right|, m > 0, n > 0 \quad (3.18)$$

Table 3.11 shows the percentages in terms of the reference prices  $\hat{P}^*$  of the MAD and the MAVRD of the (Z)CCB prices from different loss models ( $MAD_A, MAVRD_A$ ) and from the algorithm using 100 simulations of 1000 trajectories ( $MAD_B, MAVRD_B$ ), with respect to expiration time  $T$  and threshold level  $D$ . We find that most of the percentages of the  $MAD$  are similar (the difference is less than 1%) meaning no significant influence of the loss models on the (Z)CCB prices.

An explanation of the previous results is the quality of the data of losses caused by earthquakes, where 88% of the original data is missing. In our data analysis, the expected loss is considerably more important for the (Z)CCB prices than the entire distribution of losses. This is due to the nonlinear character of the modelled loss function and the dependence of different variables that affect the price of the (Z)CCB. Figure 3.12 presents the (Z)CCB prices with respect to the threshold level  $D$ , under the Burr and

### 3 Catastrophe (CAT) Bonds

	T	D	$\hat{P}^*$	(%) $MAD_A$	(%) $MAD_B$	(%) $MAVRD_A$	(%) $MAVRD_B$
ZCCB	1	100	148.576	0.283	0.975	0.329	0.265
	1	120	149.637	0.203	0.663	0.270	0.228
	1	135	149.637	0.619	0.802	0.619	0.183
	2	100	133.422	1.577	2.334	1.577	0.566
	2	120	137.439	0.823	1.306	0.823	0.375
	2	135	138.873	0.884	1.161	0.930	0.358
	3	100	114.866	4.666	5.316	4.666	0.859
	3	120	123.177	2.409	2.958	2.409	0.640
	3	135	125.766	2.468	2.817	2.468	0.520
CCB	1	100	151.236	0.513	1.152	0.556	0.257
	1	120	152.306	0.398	0.853	0.419	0.216
	1	135	152.920	0.383	0.601	0.405	0.178
	2	100	139.461	0.966	2.131	0.966	0.475
	2	120	142.950	0.731	1.585	0.774	0.395
	2	135	145.141	0.337	0.827	0.556	0.354
	3	100	124.831	2.412	3.421	2.412	0.823
	3	120	131.508	1.844	2.590	1.844	0.708
	3	135	134.324	2.071	2.474	2.071	0.600

Table 3.11: Percentages in terms of  $\hat{P}^*$  of the mean of the absolute differences and the mean of the absolute values of the relative differences of the (Zero) Coupon CAT bond prices (Z)CCB for different loss models ( $MAD_A, MAVRD_A$ ) and the (Z)CCB prices from the algorithm using one hundred simulations of 1000 trajectories ( $MAD_B, MAVRD_B$ ) with respect to expiration time  $T$  and threshold level  $D$ .

### 3 Catastrophe (CAT) Bonds

Pareto distribution for different expected loss models. The bond prices are more dispersed under different expected loss models with the same distribution assumption than under different distribution assumptions with the same loss model.

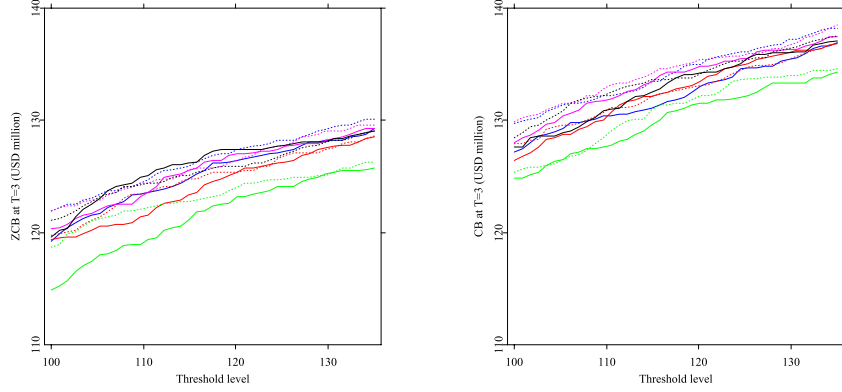


Figure 3.12: The (Zero) Coupon CAT bond prices ((left) right panel) at time to maturity  $T = 3$  years with respect to the threshold level  $D$ . The CAT bond prices under the Burr distribution (solid lines), the Pareto distribution (dotted lines) and under different loss models (different color lines).

The relevance of the modelled-index loss trigger mechanism is that it reduces the basis and moral risk borne by the sponsor and that it considers different variables that influence the underlying risk. For a given severity and frequency of earthquake risk, this analysis may be useful in determining how a CAT bond will be priced relative to an *expected level*.

### 3.4 Conclusion

The occurrence of disfavoured extreme natural events like earthquakes, hurricanes, long cold winter, heat, drought, freeze, etc. may cause substantial financial losses. In the presence of this, many sponsor companies have turned to the capital markets to cover costs of potential catastrophes by issuing CAT bonds that passes the risk on to investors.

The calibration of parametric CAT bonds for earthquakes in Mexico delivers several policy-relevant findings, e.g. on the relative costs of reinsurance and CAT bonds mixes or the inherent default risk of CAT bonds. The decision of the government to issue a parametric CAT bond relies on the fact that it triggers immediately when an earthquake meets the defined physical parameters. The parametric CAT bond especially helps the government with fast emergency services and rebuilding after a big earthquake. The calibration of the bond is based on the estimation of the intensity rate that describes the flow process of events (earthquakes that trigger the CAT bond's payoff) from the

### 3 Catastrophe (CAT) Bonds

reinsurance market, from the capital markets and from the historical data. Under the risk neutral world, the implied risk trigger rates computed from the reinsurance and capital markets lie inside the confidence intervals of the physical trigger rate, meaning that the direct access of the CAT bond into capital markets expand the risk bearing capacity beyond the limited capital held by reinsurers. The results indicate that in the presence of catastrophe risk, the default risk is also substantial for the valuation of the reinsurance premium. Therefore a combination of reinsurance and CAT bond is optimal in the sense that it provides coverage for a lower cost and lower exposure at default than reinsurance itself.

In order to reduce the basis and moral risk borne by the sponsor and to reflect the value of the loss by several variables, we price a modelled-index CAT bond (a hybrid type) for earthquakes by means of a compound doubly stochastic Poisson process. In the parametric CAT bond, the spread rate is reflected by the intensity rate of the earthquake process, while for the modelled loss trigger it is represented by the intensity rate of the earthquake process and the level of accumulated losses  $L_s$ . The trigger mechanism matters for the CAT bond pricing as long as the risk of the underlying is adequately estimated. Without doubt, the availability of information and the quality of the data provided by research institutions attempting earthquakes has a direct impact on the accuracy of this risk analysis and for the evaluation of CAT bonds.

## 4 Weather Derivatives

*How business guard against bad weather.*  
The Economist, July 12th 2008

Weather Derivatives (WD) were developed to hedge against the random nature of temperature variations that constitute weather risk. WD are financial contracts, whose payments are based on weather-related measurements. Since the first over the counter (OTC) weather derivative traded in 1997, followed in 1999 by the formal exchange Chicago Mercantile Exchange (CME) introducing derivative contracts on temperature indices. Both exchange traded and OTC are now written on a range of weather indices, including temperature, hurricanes, frost and precipitation. According to the Weather Risk Management Association, the WD market increased notably from 2.2 billion USD in 2004 to 15 billion USD through March 2009. This increase is due to the need of many institutions (energy, utility - insurance, agriculture, construction, retail, tourism - leisure industries) to hedge their exposure to weather risk and protect revenues. Since electricity and gas consumption are high temperature dependent, energy companies are one of the most important players in the weather derivative market. This chapter therefore focuses on temperature derivatives, which constitute the majority of trading volume.

WD differs from insurance, first because insurance covers low probability extreme events, for example earthquake damages, whereas WD cover lower risk high probability events such as cold winters. Second, a buyer of a WD will receive its payoff at settlement period no matter the loss caused by weather conditions. In the insurance the payoff depends on the proof of damages and indeed reduced demand is not covered.

The key factor in efficient usage of WD is a reliable valuation procedure. However, due to their specific nature one encounters several difficulties. Firstly, WD are different from most financial derivatives because the underlying weather (and indices) is not tradable or storable asset and has dynamics determined by nature not by human influence. Secondly, the weather derivative market is incomplete, meaning that the weather derivative cannot be cost-efficiently replicated by other weather derivative. Furthermore, the market is relatively illiquid. Campbell and Diebold [2005] argued that this illiquidity is due to local-specification and non-standardisation of the weather. In section 4.2.5 and 4.2.6 we show that independently of the chosen location and by using a local linear estimator for volatility, the driving stochastics are close to a Wiener Process that will allow us to do an adequate derivative pricing and hedging.

This chapter is structured as follows: the next section presents the fundamentals of temperature index derivatives (futures and options) and describes the monthly temperature futures traded at CME. In section 4.2, some properties of the temperature data are presented together with a review of the stochastic models for average daily tem-



perature available in the literature. At the end of that section an empirical analysis is conducted to explain the dynamics of temperature data for different cities. The studied temperature model captures linear trend, seasonality, mean reversion, long memory (strong autocorrelation) and seasonal volatility effects. Since temperature markets are mean reverted process explained by the conservation of energy, this thesis concentrates in Ornstein-Uhlenbeck (OU) models that model mean reversion in a natural way. On the temperature derivative market, modelling temperature volatility is an important issue for pricing and hedging. In section 4.2.5 and 4.2.6, it is demonstrated that a local linear smoothing approach corrects for seasonality, however the smoothing is not flexible enough to generally describe the volatility process well. Therefore, in this section, a local adaptive modeling approach is proposed to find at each time point, an optimal smoothing parameter to locally estimate the volatility.

Section 4.3, - the financial mathematics part - connects the weather dynamics with the pricing methodology. In section 4.4, using real data, the inverse problem of determining the MPR is solved and give (statistical and economic) interpretations of the estimated MPR. Section 4.5 shows the relationship between MPR and risk premia. We specify the MPR by introducing a new change measure and give insights of the temperature future curve. Since market participants are affected by weather risk at different locations, section 4.6 investigates the idea of a basket index on temperature at several cities. The section 4.7 concludes the chapter. This chapter follows the arguments presented in Haerdle and López-Cabrera [2010b], Benth et al. [2010] and Haerdle et al. [2010].

### 4.1 Definitions

The largest portion of futures and options written on temperature indices is traded on the CME. Most of the temperature derivatives are written on daily average temperature indices, rather than the underlying temperature by itself. Temperature futures are contracts written on different temperature indices measured over specified periods  $[\tau_1, \tau_2]$  like weeks, months or quarters of a year. The owner of a call option written on futures  $F_{(t, \tau_1, \tau_2)}$  with exercise time  $t \leq \tau_1$  and measurement period  $[\tau_1, \tau_2]$  will receive  $\max \{F_{(t, \tau_1, \tau_2)} - K, 0\}$ . The most common weather indices on temperature are: Heating Degree Day (HDD), Cooling Degree Day (CDD), Cumulative Averages (CAT). The HDD index measures the temperature over a period  $[\tau_1, \tau_2]$ , usually between October to April, when heating demands are high:

$$HDD(\tau_1, \tau_2) = \int_{\tau_1}^{\tau_2} \max(c - T_u, 0) du \quad (4.1)$$

where  $c$  is the baseline temperature (usually 18°C or 65°F) and  $T_u = (T_{u, \max} + T_{u, \min})/2$  is the average temperature on day  $u$ . Note that the temperature is a continuous-time process, even though the underlying indices are discretely monitored. Similarly, the CDD index measures the temperature over a period  $[\tau_1, \tau_2]$ , usually between April to

#### 4 Weather Derivatives

October, when cooling demands are high:

$$CDD(\tau_1, \tau_2) = \int_{\tau_1}^{\tau_2} \max(T_u - c, 0) du \quad (4.2)$$

with  $T_u = (T_{u,max} + T_{u,min})/2$  is also the average temperature on day  $u$ . The HDD and the CDD index are used to trade futures and options in 20 US cities, (Cincinnati, Colorado Springs, Dallas, Des Moines, Detroit, Houston, Jacksonville, Kansas City, Las Vegas, Little Rock, Los Angeles, Minneapolis-St. Paul, New York, Philadelphia, Portland, Raleigh, Sacramento, Salt Lake City, Tucson, Washington D.C), six Canadian cities (Calgary, Edmonton, Montreal, Toronto, Vancouver and Winnipeg) and three Australian cities (Brisbane, Melbourne and Sydney).

The CAT index accounts the accumulated average temperature over  $[\tau_1, \tau_2]$ :

$$CAT(\tau_1, \tau_2) = \int_{\tau_1}^{\tau_2} T_u du \quad (4.3)$$

where  $T_u = (T_{u,max} + T_{u,min})/2$ . The CAT index is the substitution of the CDD index for nine European cities (Amsterdam, Essen, Paris, Barcelona, London, Rome, Berlin, Madrid, Oslo, Stockholm). Since  $\max(T_u - c, 0) - \max(c - T_u, 0) = T_u - c$ , we get the HDD-CDD parity

$$CDD(\tau_1, \tau_2) - HDD(\tau_1, \tau_2) = CAT(\tau_1, \tau_2) - c(\tau_2 - \tau_1) \quad (4.4)$$

Therefore, it is sufficient to analyze only HDD and CAT indices. An index similar to the CAT index is the Pacific Rim Index, which measures the accumulated total of 24-hour average temperature (C24AT) over a period  $[\tau_1, \tau_2]$  days for Japanese Cities (Tokyo, Osaka and Hiroshima):

$$C24AT(\tau_1, \tau_2) = \int_{\tau_1}^{\tau_2} \tilde{T}_u du \quad (4.5)$$

where  $\tilde{T}_u = \frac{1}{24} \int_1^{24} T_{u_i} du_i$  and  $T_{u_i}$  denotes the temperature of hour  $u_i$ . A difference between the CAT and the C24AT index is that the latter is traded over the whole year, while the CAT index is just traded during summer.

The options at CME are cash settled i.e. the owner of a future receives 20 times the Degree Day Index at the end of the measurement period, in return for a fixed price. The currency is British pounds for the European Futures contracts, US dollars for the US contracts and Japanese Yen for the Asian cities. The minimum price increment is one Degree Day Index point. The degree day metric is Celsius and the termination of the trading is two calendar days following the expiration of the contract month. The Settlement is based on the relevant Degree Day index on the first exchange business day at least two calendar days after the futures contract month. The accumulation period of each CAT/CDD/HDD/C24AT index futures contract begins with the first calendar day of the contract month and ends with the calendar day of the contract month. Earth

Satellite Corporation reports to CME the daily average temperature. Traders bet that the temperature will not exceed the estimates from Earth Satellite Corporation. At the CME, the measurement periods for the different temperature indices are standardized to be each month of the year and seasonal strips HDD or CDD/CAT indices (minimum of two, and maximum of seven consecutive calendar months).

In this chapter the study concentrates on monthly contracts, but similar implications can be done for seasonal strip contracts. Let us simplify the notation by writing different contracts  $i = 1, \dots, I$  traded at time  $t$  with measurement period  $t \leq \tau_1^i < \tau_2^i$ . Therefore a contract with  $i = 7$  is six months ahead from the trading day  $t$ . For US and Europe CAT/CDD/HDD futures  $I$  is usually equal to 7 (April-November or November-April), while for Asia  $I = 12$  (Jan-Dec).

## 4.2 Modelling Temperature

In this section, the stochastic process that model the daily average temperature, describe temperature fluctuations and allow analytical pricing of contracts are discussed.

### 4.2.1 Properties of temperature data

Temperature is a continuous time process even though the indices used as underlying for temperature futures contracts are discretely monitored. Most of the literature that discuss models for daily average temperature capture a linear trend, seasonality, mean reversion, seasonality volatility. Daily average temperature reflects not only a seasonal pattern from calendar effects (peaks in cooler winter and warmer summers) but also a variation that varies seasonally. Many of the empirical analysis results have shown that temperatures for American and Europe cities have a binomial distribution. Temperature behaves around a normal level (mean reversion) and is persistent, with pronounced cyclical dynamics and strong correlations (long memory). Campbell and Diebold [2005], Cao and Wei [2004] and Alaton et al. [2002] have discussed the issue of global warming by detecting a linear trend in the temperature data. Related to this phenomena are the urban heating effect and the air pollution, which increase the global temperature, Campbell and Diebold [2005].

Since temperature markets are mean reverted process explained by the conservation of energy, the study concentrates on Ornstein-Uhlenbeck (OU) models that model mean reversion in a natural way. Most of the models in the literature focused on mean reverted models for temperature differ from their definition of temperature variations, which is exactly the component that characterizes weather risk. The behaviour of this component under a fractional Brownian motion, a Lévy Process and a standard Brownian Motion will be studied. Then an empirical analysis using a multidimensional OU process is conducted to real data. For a particular location, let:

1.  $T_t$  be the average temperature on day  $t$ .
2.  $\Lambda_t$  be a bounded and deterministic function, denotes the seasonal effect and it is the mean reversion level of temperature at day  $t$ .

3.  $\alpha_t$  be a bounded and deterministic function, reflecting the rate of mean reversion at time  $t$ .
4.  $\sigma_t$  be a bounded and deterministic function, representing the seasonal variation of daily average temperature at time  $t$ .

#### 4.2.2 An Ornstein-Uhlenbeck driven by a Fractional Brownian Motion

Suppose that the process  $T_t$  is modelled as in Brody et al. [2002]:

$$dT_t = -\alpha_t (T_t - \Lambda_t) dt + \sigma_t dB_t^H \quad (4.6)$$

where  $B_t^H$  is a fractional Brownian Motion (FBM) defined on the probability space  $(\Omega, \mathcal{F}, \mathbb{P}^H)$  and  $H \in (0, 1)$  denotes the Hurst parameter.  $B_t^H$  is a continuous time Gaussian process with zero mean and starts at zero ( $B_0^H = 0$ ). Under the measure  $\mathbb{P}^H$  the correlation function is defined as:

$$\mathbb{E}^H [B_t^H B_s^H] = \frac{1}{2} (t^{2H} + s^{2H} - |t - s|^{2H}), \forall t, s \geq 0 \quad (4.7)$$

When  $H = 0.5$ , the correlation between increments is zero and therefore the FBM is a standard Brownian Motion. If  $H > 0.5$  the correlation is negative and positive when  $H < 0.5$ .

Brody et al. [2002] argued that this model can capture the mean time dependent level, the normality and persistence in temperature, specially when  $H > 0.5$  the temperature process exhibits long memory dependence since:

$$\sum_{n=1}^{\infty} \mathbb{E}^H [B_1^H B_{n+1}^H - B_n^H] = \infty \quad (4.8)$$

They propose to use a Hurst exponent  $H = 1 + \alpha, \alpha \leq 0$ . They estimated from fluctuations after temperature data has been deseasonalized a  $H > \frac{1}{2}$  with  $\alpha = -0.39$ . Consequently,  $T_t$  is a normal distributed random variable with mean:

$$\mathbb{E}^H [T_t] = \exp \left( \int_0^t \alpha_s ds \right) \left[ T_0 + \int_0^t \alpha_s \Lambda_s \left\{ \exp \left( \int_0^t \alpha_s ds \right) \right\}^{-1} ds \right] \quad (4.9)$$

and variance:

$$\begin{aligned} \text{Var} [T_t] = & \left\{ \exp \left( \int_0^t \alpha_s ds \right) \right\}^2 \int_0^t \int_0^t \sigma_u \sigma_s \left\{ \exp \left( \int_0^t \alpha_u du \right) \right\}^{-1} \left\{ \exp \left( \int_0^t \alpha_s ds \right) \right\}^{-1} \\ & \times H(2H - 1) |u - s|^{2H-2} du ds \end{aligned} \quad (4.10)$$

Benth and Benth [2005] in this framework proposed a consistent model to correct mean

## 4 Weather Derivatives

reversion model:

$$dT_t = d\Lambda_t - \alpha_t (T_t - \Lambda_t) dt + \sigma dB_t^H \quad (4.11)$$

Although FBM captures persistence effects, it is not a semimartingale, which is a requirement to work under the incomplete market setting. Benth [2003] derived no arbitrage prices of (4.11) using quasi-conditional expectations and fractional stochastic calculus. There is a big discussion in the literature about the arbitrage opportunities of this model.

### 4.2.3 An Ornstein-Uhlenbeck Model driven by a Brownian motion

Let  $B_t$  be a Brownian motion on the probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  in the stochastic process

$$dT_t = d\Lambda_t - \alpha (T_t - \Lambda_t) dt + \sigma_t dB_t \quad (4.12)$$

with explicit dynamic for the daily average temperature given by:

$$T_t = \Lambda_t + \{T_0 - \Lambda_0\} \exp\{-\alpha t\} + \int_0^t \sigma_u \exp\{-\alpha(t-u)\} dB_u \quad (4.13)$$

Alaton et al. [2002] used (4.13) to analyse forty years of Swedish temperature data. They proposed a monthly varying volatility function  $\sigma$  and observed that daily temperature differences were very close to normality. Similarly, Benth and Benth [2007] fitted Sweden daily average temperatures. They founded a strong seasonal variation in the autocorrelation of squared daily residuals after detrending and deseasonalizing daily temperatures and modelling the mean reverting property with an autoregressive process  $AR(1)$ . After removing the seasonal variation, standardized residuals were close to normality, with rejection at 1% significance level. However, the assumption of independent identically distributed residuals is enough to calculate temperature derivatives. Benth et al. [2007b] approximates also the noise with a Brownian motion in a  $n$ -dimensional setting, allowing analytical derivative prices.

### 4.2.4 An Ornstein-Uhlenbeck Model driven by a Lévy Process

Benth and Benth [2005] analysed temperature data in Norway cities by using the discrete time version of a non-Gaussian-Uhlenbeck model:

$$dT_t = d\Lambda_t - \alpha (T_t - \Lambda_t) dt + \sigma_t dL_t \quad (4.14)$$

which has the form of additive time series:

$$T_t = \Lambda_t + c_t + \tilde{\varepsilon}_t, t = 0, 1, 2, \dots \quad (4.15)$$

where  $L_t$  is a Lévy process and  $c_t$  is the cyclical component and the random noise  $\tilde{\varepsilon}_t$ . The residuals obtained after dividing by the square root of the seasonal variance show

heavy tails, meaning that a Lévy process is a good candidate to model generalised hyperbolic distributions (distributions with known density and characteristic functions). By applying Itô's Lemma, the explicit solution to (4.14) is:

$$T_t = \Lambda_t + (T_0 - \Lambda_0) \exp(-\alpha t) + \int_0^t \sigma_u \exp\{-\alpha(t-u)\} dL_u \quad (4.16)$$

In order to price temperature derivatives, the authors used the Esscher transformation to specify probability measures. They argued that FBM is not appropriate for modelling Norwegian temperature data and suggest to use a regression model with deseasonalized temperature with a seasonal variance.

More study should be carried into more models that compromise between modelling flexibility and analytical tractability for computing derivative prices. A time series approach is conducted in the section 4.2.5, to model the daily average temperature in Europe, America and Asia. The seasonality and volatility vary for every of the studied cities in amplitude and pattern. This highlights the importance of flexibility of the model seasonality specific to a certain location. Low order Fourier truncated series with GARCH models and local linear regression estimator describe well the seasonal and variation components.

### 4.2.5 Empirical Analysis of Temperature Dynamics

In the next section, a continuous autoregressive model of high order is proposed to model daily average temperatures rather than the underlying temperature itself, in order to derive explicitly no arbitrage prices for temperature future and options. The thesis concentrates on the modelling of this quantity, because most of the temperature derivative trading is based on average daily temperature. Due to no availability of data and for simplicity, intra daily temperature effects are not taken into account.

This chapter studies the average daily temperature data for US, Europe and Asian Cities. In particular, the thesis analyses weather dynamics for Atlanta, Portland, Houston, New York, Berlin, Essen, Tokyo, Osaka, Beijing and Taipei. Our interest in these cities is first, because all of them with the exception of the latter two are traded at CME, second, because a casual examination of the trading statistics on the CME website reveals that the Atlanta HDD, Houston CDD and Portland CDD temperature contracts have relatively more liquidity and third, because we can infer the MPR for regions without weather derivative markets knowing the formal dependence of MPR on seasonal variation.

The presence of a linear trend is first checked and the seasonal pattern  $\Lambda_t$  of the daily temperatures  $T_t$  ( $X_t = T_t - \Lambda_t$ ) is investigated. The seasonality of different locations is modelled by using the next least squares fitted seasonal function with trend:

$$\Lambda_t = a + bt + \sum_{l=1}^L c_l \cos \left\{ \frac{2\pi(t - d_l)}{l \cdot 365} \right\} \quad (4.17)$$

#### 4 Weather Derivatives

City (Period)	$\hat{a}$ (CI)	$\hat{b}$ (CI)	$\hat{c}_1$ (CI)	$\hat{d}_1$ (CI)
Portland (19480101-20081204)	55.35 (55.35,55.36)	-0.0116 (-0.0166,-0.0065)	14.36 (14.36,14.37)	-155.58 (-155.58,-155.57)
Atlanta (19480101-20081204)	61.95 (61.95,61.96)	-0.0025 (-0.0081,0.0031)	18.32 (18.31,18.33)	-165.02 (-165.03,-165.02)
New York (19490101-20081204)	53.86 (53.86,53.87)	-0.0004 (-0.0079,0.0071)	21.43 (21.42,21.44)	-156.27 (-156.27,-156.26)
Houston (19700101-20081204)	68.52 (68.51,68.52)	-0.0006 (-0.0052,0.0039)	15.62 (15.62,15.63)	-165.78 (-165.79,-165.78)
Berlin (19480101-20080527)	9.72 (9.71,9.74)	-0.0004 (-0.0147,0.0139)	9.75 (9.74,9.77)	-164.79 (-164.81,-164.78)
Essen (19700101-20090731)	10.80 (10.79,10.81)	-0.0020 (-0.0134,0.0093)	8.02 (8.01,8.03)	-161.72 (-161.73,-161.71)
Tokyo (19730101-20090831)	16.32 (16.31,16.33)	-0.0003 (-0.0085,0.0079)	10.38 (10.37,10.38)	-153.52 (-153.53,-153.52)
Osaka (19730101-20090604)	16.78 (16.77,16.79)	-0.0021 (-0.0109,0.0067)	11.61 (11.60,11.62)	-153.57 (-153.58,-153.56)
Beijing (19730101-20090831)	12.72 (12.71,12.73)	0.0001 (-0.0070,0.0073)	14.93 (14.92,14.94)	-169.59 (-169.59,-169.58)
Taipei (19920101-20090806)	23.32 (23.31,23.33)	0.0023 (-0.0086,0.0133)	6.67 (6.66,6.68)	-158.67 (-158.68,-158.66)

Table 4.1: Coefficients of the Fourier truncated seasonal series of average daily temperatures in different cities. All coefficients are nonzero at 1% significance level. Confidence intervals are given in parenthesis.

where the coefficients  $a$  and  $b$  indicate the average temperature and global Warming respectively. We observe low temperatures in winter times and high temperatures in the summer for different locations. The temperature data sets do not deviate from its mean level and a linear trend at 1% significance level is detectable. The corresponding estimates in most of the cities (0.0001) translates to a change of 1.825 degrees ( $50 \times 365 \times 0.0001$ ) in a 50 year span, which is a trivial change as far as global warming is concerned, see Table 4.1. Our findings are similar to Alaton et al. [2002], Campbell and Diebold [2005], Benth et al. [2007b], Benth et al. [2007a] for Sweden, USA and Lithuania respectively.

Note that the series expansion in (4.17) with more and more periodic terms provides a fine tuning but this will increase the number of parameters. Here we propose a different way to correct for seasonality. We show that a local smoothing approach does that job. Asymptotically they can be approximated by Fourier series estimators.

The data is smoothed with a Local Linear Regression (LLN)  $\Lambda_{t,LLN}$  estimator:

$$\arg \min_{e,f} \sum_{t=365}^1 \{ \bar{T}_s - e_s - f_s(t-s) \}^2 K\left(\frac{t-s}{h}\right) \quad (4.18)$$

where  $\bar{T}_s$  is the mean of average daily temperature in year  $j = 1 \dots n$  and  $K(\cdot)$  is a kernel. This estimator incorporates an asymmetry term since the difference of temper-

#### 4 Weather Derivatives

atures in winter is more pronounced than the summer high temperatures, as Figure 4.1 displays in a stretch of eight years plot of the average daily temperatures, the Fourier truncated and the local linear seasonal component using Epanechnikov Kernel.

After removing the seasonality (4.18) from the daily average temperatures ( $X_t = T_t - \Lambda_{t,LNN}$ ), we check whether  $X_t$  is a stationary process  $I(0)$ . We apply the Augmented Dickey-Fuller test (ADF):

$$(1 - L)X = c_1 + \mu t + \tau LX + \alpha_1(1 - L)LX + \dots \alpha_p(1 - L)L^p X + \varepsilon_t$$

where  $p$  is the number of lags of  $(1 - L)X$  by which the regression is augmented to get residuals free of autocorrelation. If  $H_0 : \tau = 0$  (unit root) is rejected,  $\tau$  hence  $X_t$  is a stationary process  $I(0)$ .

This result can also be verified by using the KPSS Test:

$$X_t = c + \mu t + k \sum_{i=1}^t \xi_i + \varepsilon_t$$

with stationary  $\varepsilon_t$  and iid  $\xi_t$  with zero expected value and variance equal to one. We accept  $H_0 : k = 0$  that the process is stationary. We then plot the Partial Autocorrelation Function (PACF) of  $X_t$  in Figure (4.2), which suggests that persistence (pronounced cyclical dynamics and strong intertemporal correlation) of daily average is captured by autoregressive processes of higher order  $p$ :

$$X_{t+p} = \sum_{i=1}^p \beta_i X_{t+p-i} + \sigma_t \varepsilon_t \quad (4.19)$$

However, when plotting the PACF over different year-lengths moving windows, we found that the long memory diagnosis of temperature is replicated by a short memory process with structural breaks. Similar empirical analysis results are found in Diebold and Inoue [2001], Granger and Hyung [2004], Mercurio and Spokoiny [2004] and Benth et al. [2010]. We verify that the  $AR(3)$  suggested by Benth et al. [2007b] holds for many cities. Table 4.2 shows the results of the stationarity tests as well as the coefficients of the fitted  $AR(3)$ .

According to the modified Li-McLeod Portmanteau test, in most of the cases, we reject at 1% significance level the null hypothesis  $H_0$  that the residuals  $\varepsilon_t$  are uncorrelated. After trend and seasonal components were removed, the residuals  $\varepsilon_t$  and the squared residuals  $\varepsilon_t^2$  of temperature data of Equation (4.19) are plotted in Figure 4.3 and 4.4. The ACF's of the residuals of  $AR(3)$  are close to zero and according to Box-Ljung statistic the first few lags are insignificant. Nevertheless we observed a high seasonal pattern in the ACF for the squared residuals  $\varepsilon_t$ , see Figure 4.5.

The seasonal variance of residuals  $\sigma_t^2$  is calibrated with the 2 step model of Campbell



#### 4 Weather Derivatives

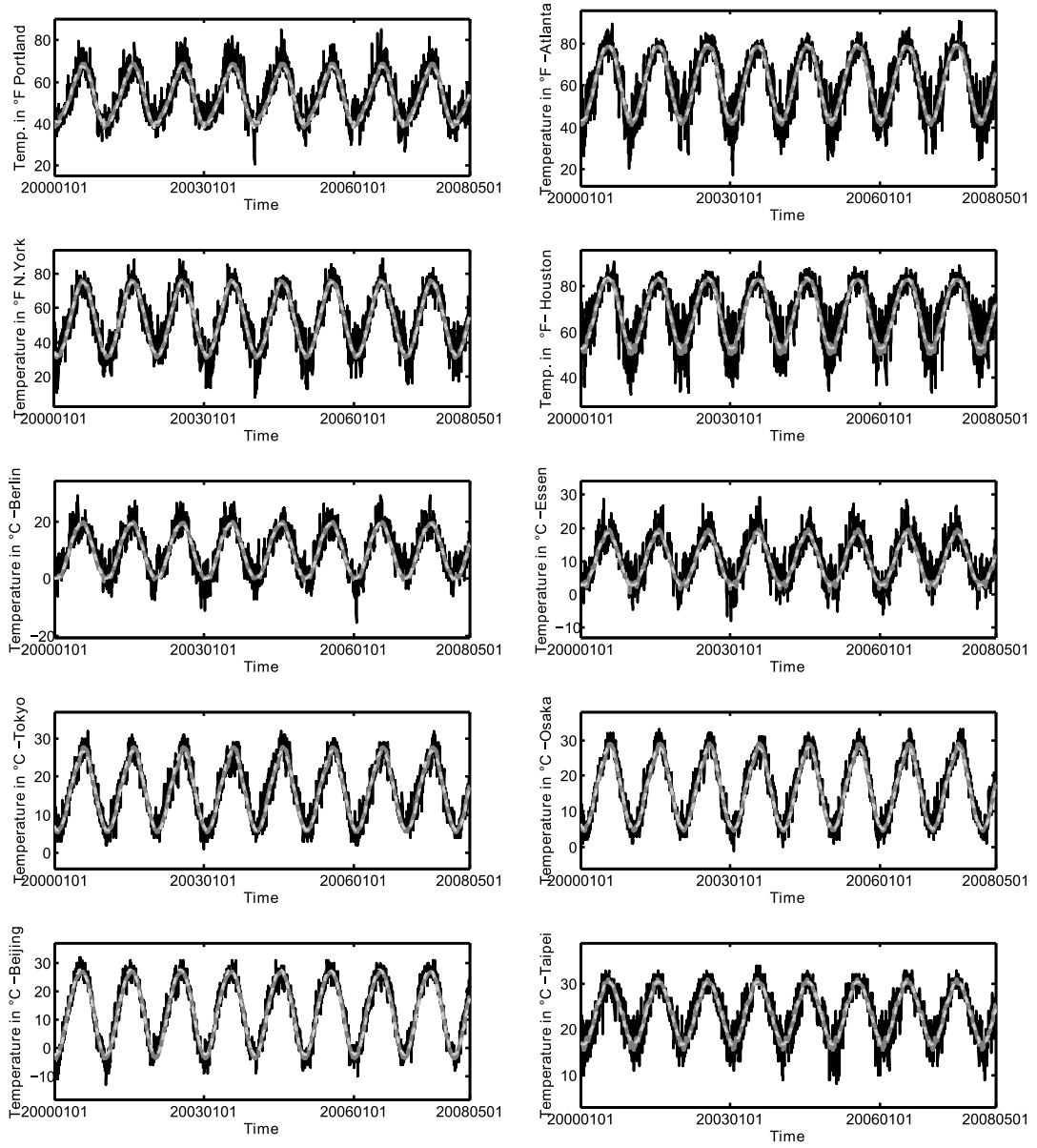


Figure 4.1: Average daily temperatures (black line), the Fourier truncated (dashed line) and the local linear seasonal (gray line) component for different cities.

## 4 Weather Derivatives

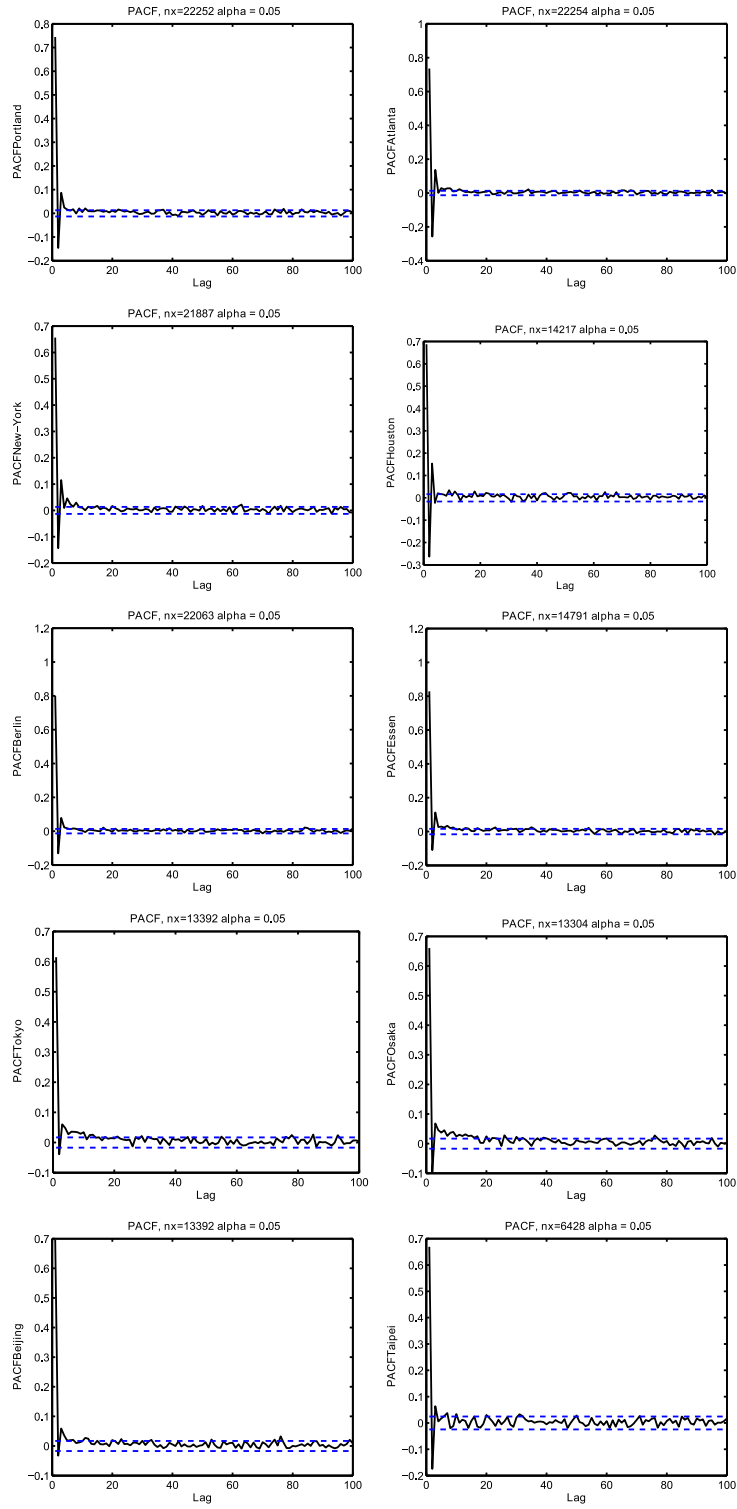


Figure 4.2: PACF of detrended temperatures for different cities.

#### 4 Weather Derivatives

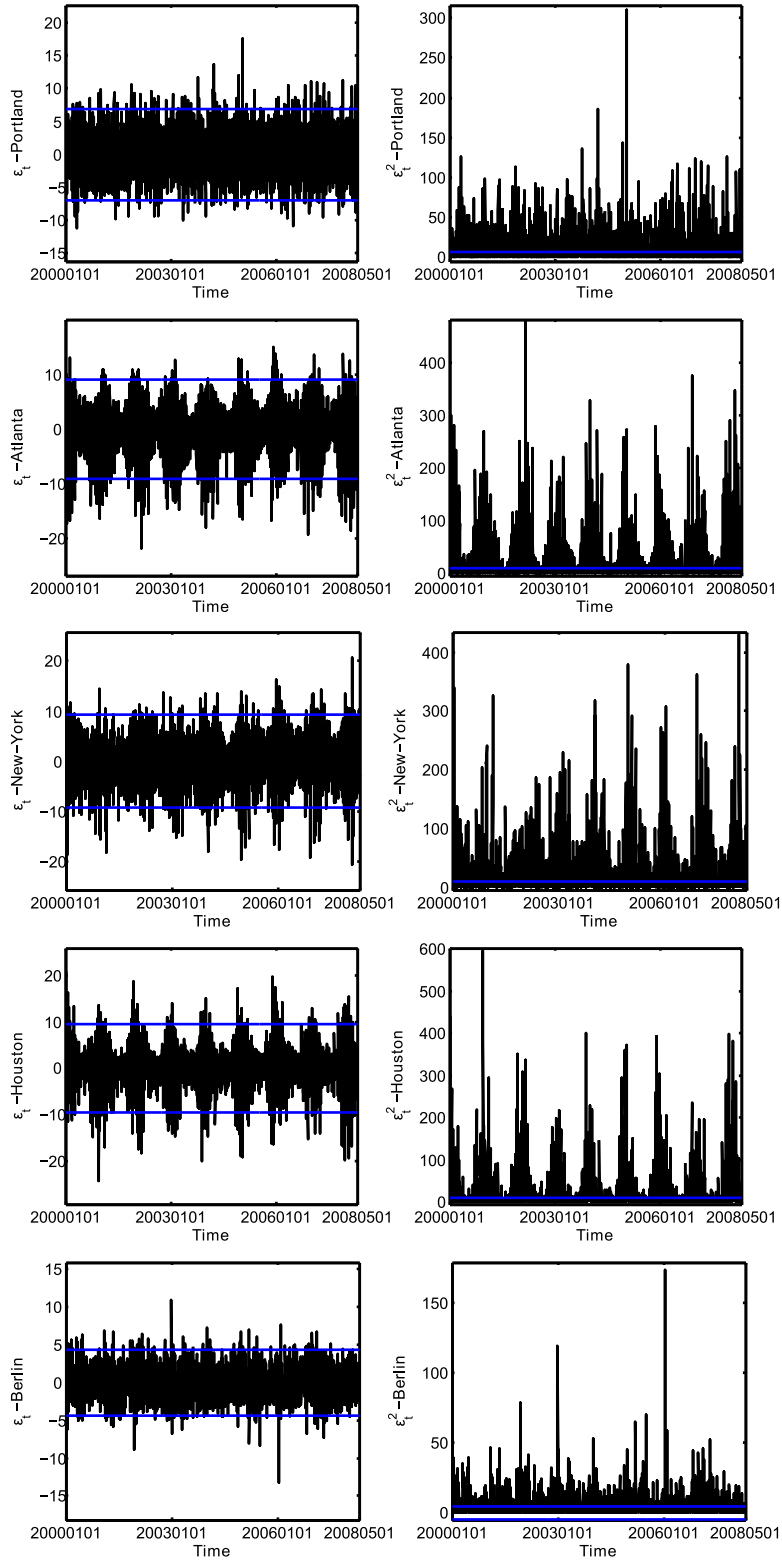


Figure 4.3: Residuals of daily temperatures  $\varepsilon_t$  (left panels), Squared residuals  $\varepsilon_t^2$  (right panels) for different cities.

#### 4 Weather Derivatives

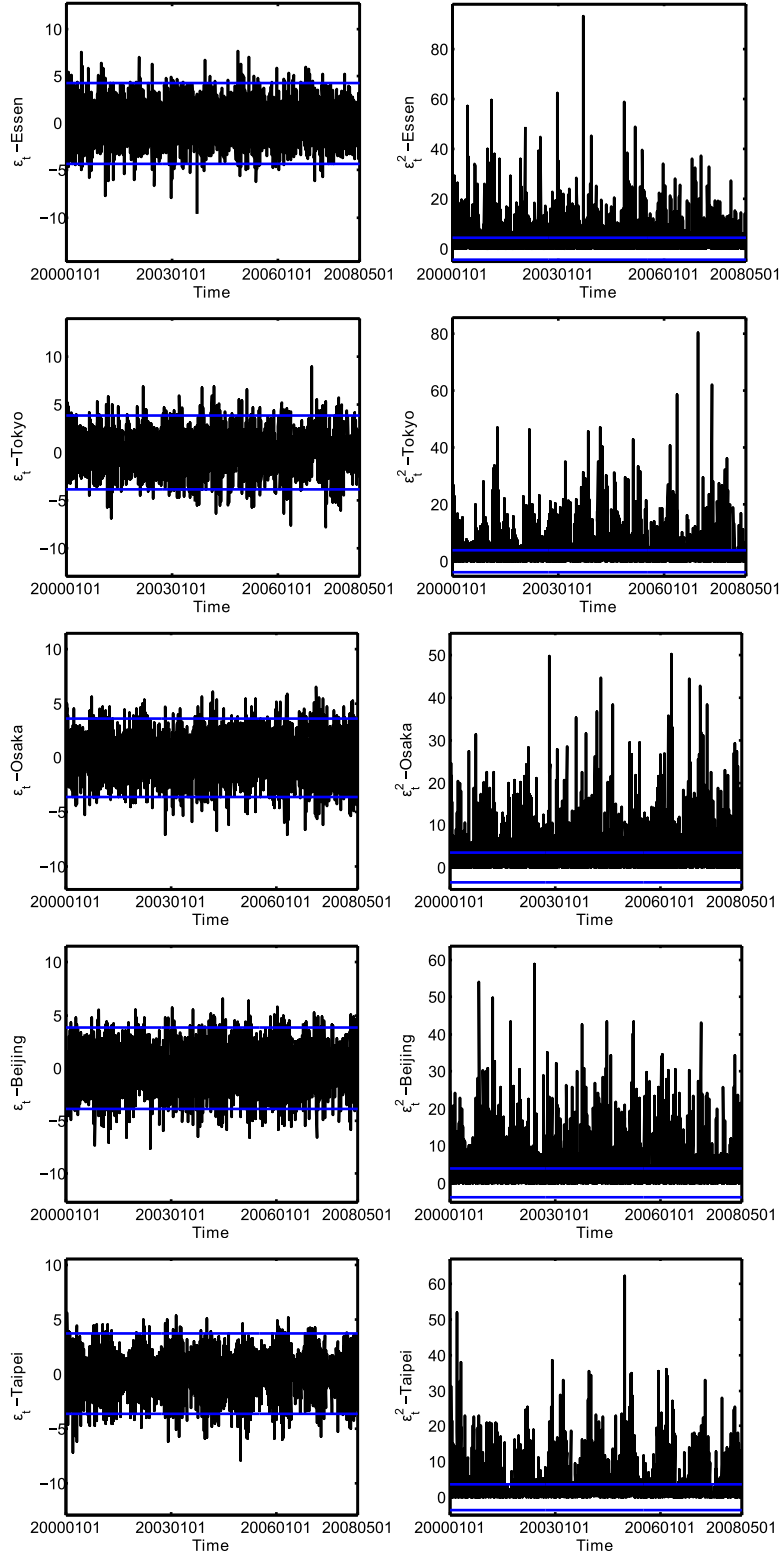


Figure 4.4: Residuals of daily temperatures  $\varepsilon_t$  (left panels), Squared residuals  $\varepsilon_t^2$  (right panels) for different cities.

## 4 Weather Derivatives

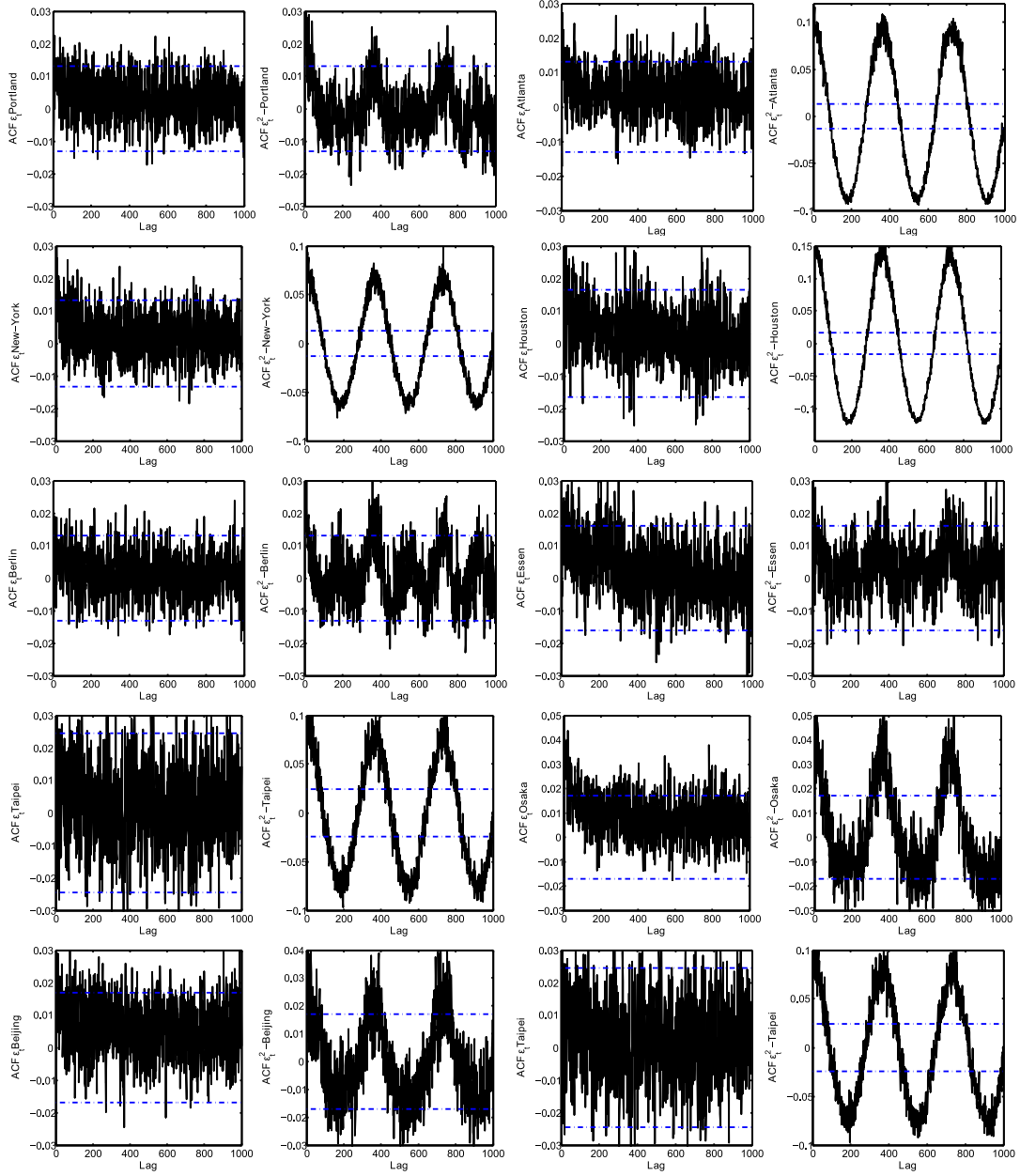


Figure 4.5: ACF of Residuals of daily temperatures  $\varepsilon_t$  (left panels), Squared residuals  $\varepsilon_t^2$  (right panels) for different cities.

#### 4 Weather Derivatives

City	ADF	KPSS	AR(3)			CAR(3)				
	$\hat{\tau}$	$\hat{k}$	$\beta_1$	$\beta_2$	$\beta_3$	$\alpha_1$	$\alpha_2$	$\alpha_3$	$\lambda_1$	$\lambda_{2,3}$
Portland	-45.13+	0.05*	0.86	-0.22	0.08	2.13	1.48	0.26	-0.27	-0.93
Atlanta	-55.55+	0.21***	0.96	-0.38	0.13	2.03	1.46	0.28	-0.30	-0.86
New York	-56.88+	0.08*	0.76	-0.23	0.11	2.23	1.69	0.34	-0.32	-0.95
Houston	-38.17+	0.05*	0.90	-0.39	0.15	2.09	1.57	0.33	-0.33	-0.87
Berlin	-40.94+	0.13**	0.91	-0.20	0.07	2.08	1.37	0.20	-0.21	-0.93
Essen	-23.87+	0.11*	0.93	-0.21	0.11	2.06	1.34	0.16	-0.16	-0.95
Tokyo	-25.93+	0.06*	0.64	-0.07	0.06	2.35	1.79	0.37	-0.33	-1.01
Osaka	-18.65+	0.09*	0.73	-0.14	0.06	2.26	1.68	0.34	-0.33	-0.96
Beijing	-30.75+	0.16***	0.72	-0.07	0.05	2.27	1.63	0.29	-0.27	-1.00
Taipei	-32.82+	0.09*	0.79	-0.22	0.06	2.20	1.63	0.36	-0.40	-0.90

Table 4.2: ADF and KPSS-Statistics, coefficients of the autoregressive process  $AR(3)$ ,  $CAR(3)$  and eigenvalues  $\lambda_{1,2,3}$ , for the daily average temperatures time series for different cities. +0.01 critical values, \* 0.1 critical value (0.11), \*\*0.05 critical value (0.14), \*\*\*0.01 critical value (0.21).

and Diebold [2005] ( $\hat{\sigma}_{t,FTSG}^2$ ):

$$\hat{\sigma}_{t,FTSG}^2 = c_1 + \sum_{l=1}^L \left\{ c_{2l} \cos\left(\frac{2l\pi t}{365}\right) + c_{2l+1} \sin\left(\frac{2l\pi t}{365}\right) \right\} + \alpha_1(\sigma_{t-1}^2 \varepsilon_{t-1})^2 + \beta_1 \sigma_{t-1}^2 \quad (4.20)$$

In order to have model flexibility as the estimator in (4.17), the calibration in terms of a Local Linear Regression  $\hat{\sigma}_{t,LLR}^2$  is proposed:

$$\arg \min_{g,h} \sum_{t=1}^{365} \left\{ \hat{\varepsilon}_t^2 - g_s - h_s(t-s) \right\}^2 K\left(\frac{t-s}{h}\right) \quad (4.21)$$

where  $K(\cdot)$  is a kernel. Figure 4.6 shows the daily empirical variance (the average of squared residuals for each day of the year), the  $\hat{\sigma}_{t,FTSG}^2$  and the  $\hat{\sigma}_{t,LLR}^2$  estimators, using Epanechnikov kernel. Here we obtain the Campbell and Diebold [2005] effect for different temperature data, high variance in winter - earlier summer and low variance in spring - late summer.

Figure 4.7 shows the ACF of temperature residuals  $\varepsilon_t$  and squared residuals  $\varepsilon_t^2$ , after dividing out the seasonal volatility  $\hat{\sigma}_{t,LLR}^2$  from the regression residuals. The ACF plot of the standardized residuals remain unchanged but now the squared residuals presents a non-seasonal pattern.

The series expansion estimator used for the seasonal variance  $\hat{\sigma}_{t,FTSG}^2$  creates almost normal residuals, but fails in the peak seasons as Figure 4.8 in a log Kernel smoothing density plot shows against a Normal Kernel evaluated at 100 equally spaced points.  $\hat{\sigma}_{t,LLR}^2$  corrects that defect. Table 4.3 shows the calibrated coefficients of  $\hat{\sigma}_{t,FTSG}^2$ ,  $\hat{\sigma}_{t,LLR}^2$  and statistics of the standardized residuals. It is exactly the random noise of tempera-

## 4 Weather Derivatives

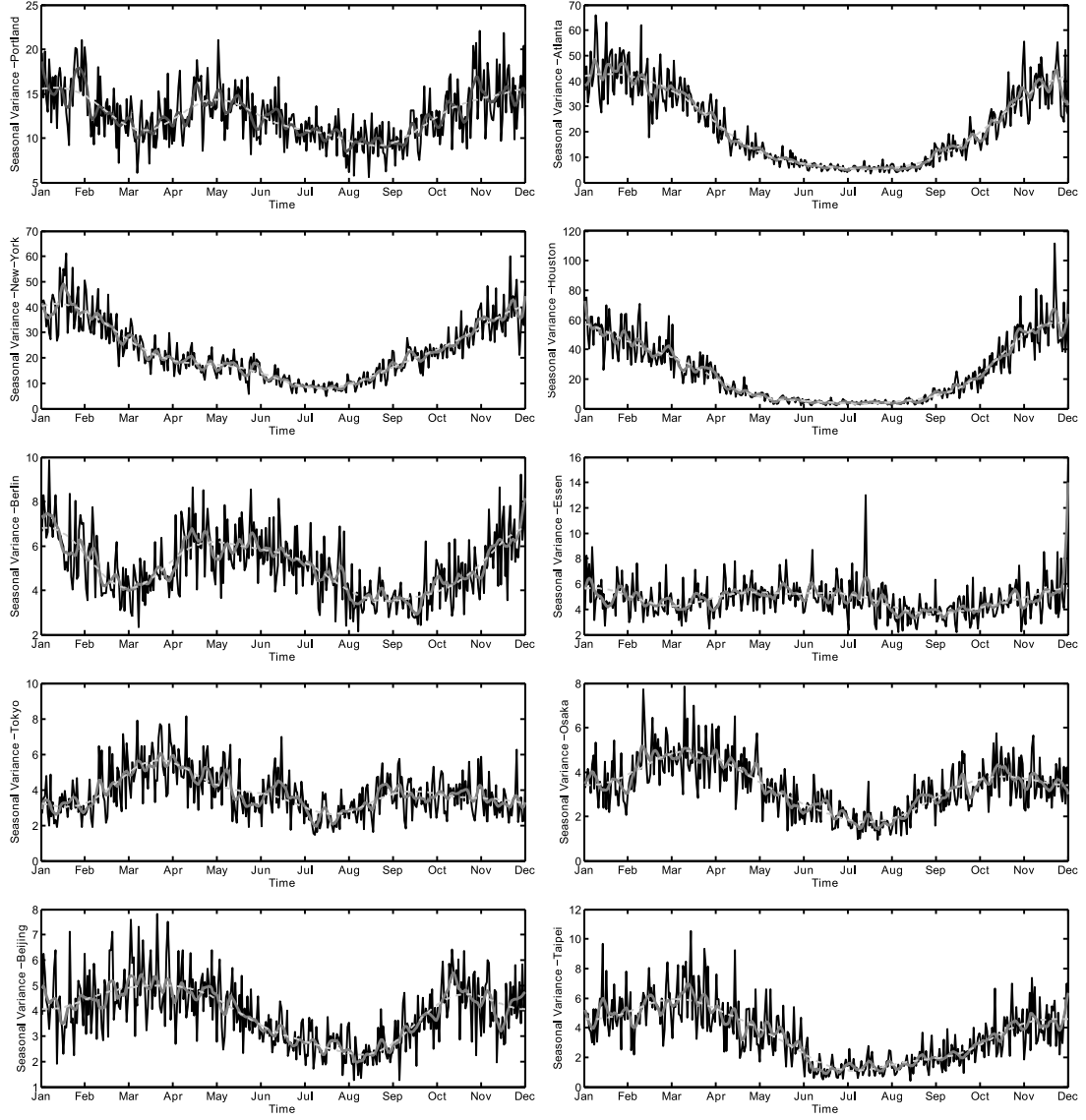


Figure 4.6: Daily empirical variance (black line), the Fourier truncated (dashed line) and the local linear smoother seasonal variation (gray line) for different cities.

## 4 Weather Derivatives

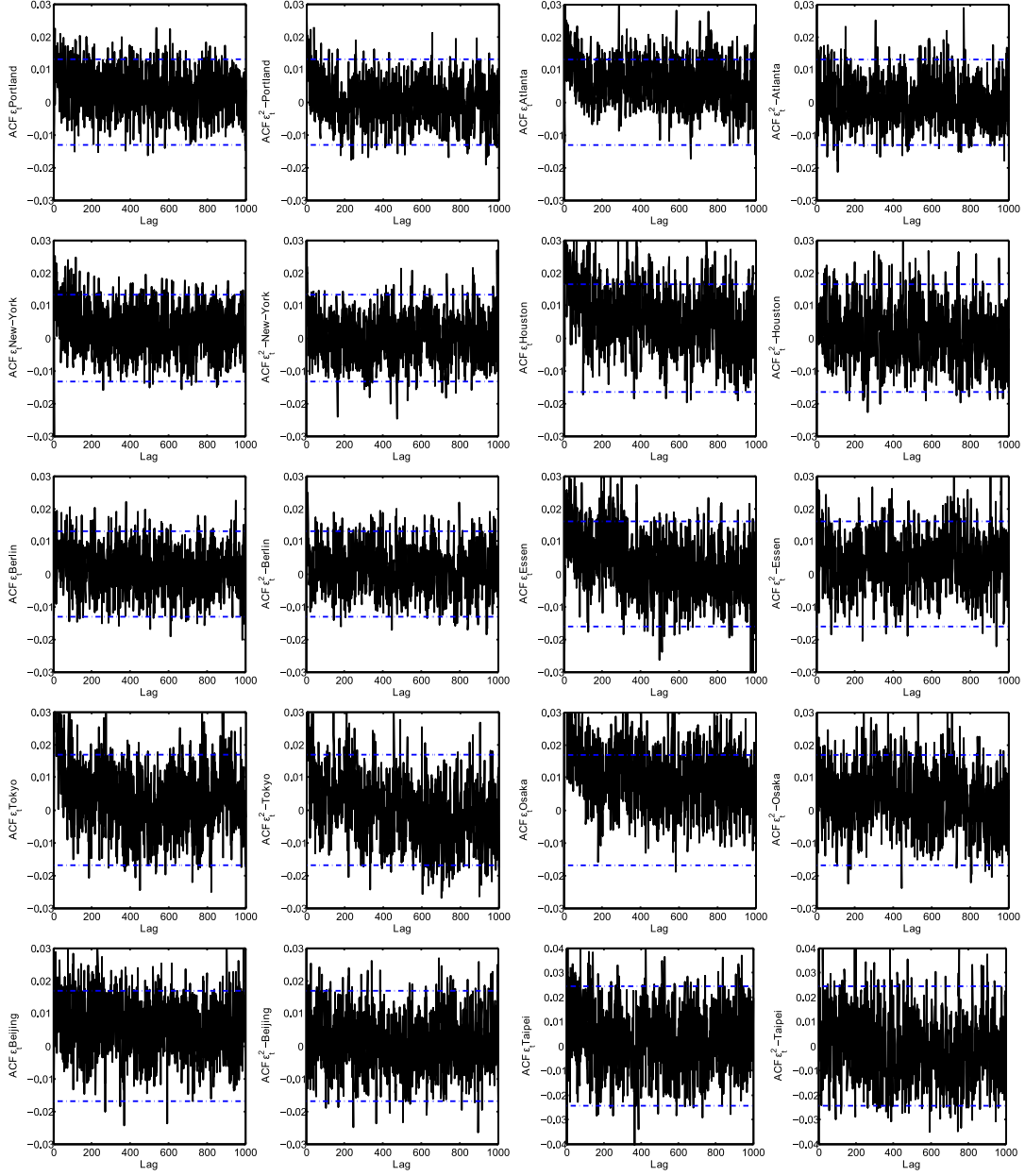


Figure 4.7: ACF of Residuals of daily temperatures  $\varepsilon_t$  (left panels), Squared residuals  $\varepsilon_t^2$  (right panels) after dividing out the seasonal volatility  $\hat{\sigma}_{t,LLR}^2$  from the regression residuals for different cities.



ture variations that constitutes weather risk. We show that independently of the chosen location, the driving stochastics are close to a Wiener Process that will allow us to do an adequate derivative pricing and hedging.

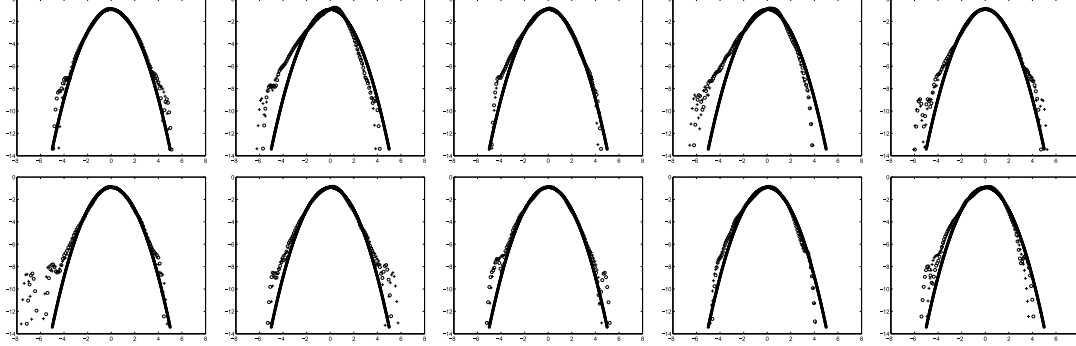


Figure 4.8: Log of Normal Kernel (stars) and Log of Kernel smoothing density estimate of standardized residuals  $\hat{\varepsilon}_t/\hat{\sigma}_{t,LLR}$  (circles) and  $\hat{\varepsilon}_t/\hat{\sigma}_{t,FTSG}$  (crosses) for different cities. From left to right upper panel: Portland, Atlanta, New York, Houston. From left to right lower panel: Berlin, Essen, Tokyo, Osaka, Beijing, Taipei.

### 4.2.6 Localizing temperature residuals

When pricing temperature derivatives, one has to deal with the residuals' dynamics of temperature  $T_t$  (temperature at time point  $T$ ). It is well known that the distribution of residuals depends on the chosen location. Benth et al. [2010] found that temperatures in many cities in Asia, Europe and America are normal in the sense that after seasonality and correlation correction, the driving stochastics are close to a Wiener Process. However, they also found that cities like Taipei and Koahsiung are characterize by different temperature types although the distance between them is very close, see Figure 4.9. For the case of these locations, the seasonal variation  $\Lambda_t$  can be seen from Figure 4.11. Leading to an approximate Normal stochastic driver, one need to incorporate an asymmetry term since the dip of temperature in winter is more pronounced than the summer high temperature. The series expansion used in (4.20) creates almost normal residuals but failed in the peak seasons as Figure 4.12 in a log density plot shows.

One may of course pursue a fine tuning of a series expansion with more and more periodic term but this will evidently increase the number of parameters. In (4.21) is demonstrated that a local linear smoothing approach corrects for seasonality, however the smoothing is not flexible enough to generally describe the volatility process well. Therefore, in this section, a local adaptive modelling approach is proposed to find at each time point, an optimal smoothing parameter to locally estimate the volatility.

The empirical work on the dynamics of temperature in Benth et al. [2010] shows similarities with the results found on realized volatility in Chen et al. [2010] or with the

#### 4 Weather Derivatives

City							Stat	$\hat{\varepsilon}_t/\hat{\sigma}_{t,FTSG}$	$\hat{\varepsilon}_t/\hat{\sigma}_{t,LLR}$
Portland	$\hat{c}_1$	12.48	$\hat{c}_2$	1.55	$\hat{c}_3$	1.05	JB	67.10	75.01
	$\hat{c}_4$	1.42	$\hat{c}_5$	-1.19	$\hat{c}_6$	0.46	Kurt	3.24	3.27
	$\hat{c}_7$	0.34	$\hat{c}_8$	-0.40	$\hat{c}_9$	0.45	Skew	0.06	0.02
Atlanta	$\hat{c}_1$	21.51	$\hat{c}_2$	18.10	$\hat{c}_3$	7.09	JB	272.01	253.24
	$\hat{c}_4$	2.35	$\hat{c}_5$	1.69	$\hat{c}_6$	-0.39	Kurt	3.98	3.91
	$\hat{c}_7$	-0.68	$\hat{c}_8$	0.24	$\hat{c}_9$	-0.45	Skew	-0.70	-0.68
New-York	$\hat{c}_1$	22.29	$\hat{c}_2$	13.80	$\hat{c}_3$	3.16	JB	367.38	355.03
	$\hat{c}_4$	3.30	$\hat{c}_5$	-0.47	$\hat{c}_6$	0.80	Kurt	3.43	3.43
	$\hat{c}_7$	2.04	$\hat{c}_8$	0.11	$\hat{c}_9$	0.01	Skew	-0.23	-0.22
Houston	$\hat{c}_1$	23.61	$\hat{c}_2$	25.47	$\hat{c}_3$	4.49	JB	140.97	122.83
	$\hat{c}_4$	6.65	$\hat{c}_5$	-0.38	$\hat{c}_6$	1.00	Kurt	3.96	3.87
	$\hat{c}_7$	-2.67	$\hat{c}_8$	0.68	$\hat{c}_9$	-1.56	Skew	-0.60	-0.57
Berlin	$\hat{c}_1$	5.07	$\hat{c}_2$	0.10	$\hat{c}_3$	0.72	JB	224.55	274.83
	$\hat{c}_4$	0.98	$\hat{c}_5$	-0.43	$\hat{c}_6$	0.45	Kurt	3.48	3.51
	$\hat{c}_7$	0.06	$\hat{c}_8$	0.16	$\hat{c}_9$	0.22	Skew	-0.05	-0.08
Essen	$\hat{c}_1$	4.78	$\hat{c}_2$	0.00	$\hat{c}_3$	0.42	JB	273.90	251.89
	$\hat{c}_4$	0.63	$\hat{c}_5$	-0.20	$\hat{c}_6$	0.17	Kurt	3.65	3.61
	$\hat{c}_7$	-0.06	$\hat{c}_8$	0.05	$\hat{c}_9$	0.17	Skew	-0.05	-0.08
Tokyo	$\hat{c}_1$	3.80	$\hat{c}_2$	0.01	$\hat{c}_3$	0.73	JB	137.93	156.58
	$\hat{c}_4$	-0.69	$\hat{c}_5$	-0.33	$\hat{c}_6$	-0.14	Kurt	3.45	3.46
	$\hat{c}_7$	-0.14	$\hat{c}_8$	0.26	$\hat{c}_9$	-0.13	Skew	-0.10	-0.13
Osaka	$\hat{c}_1$	3.34	$\hat{c}_2$	0.80	$\hat{c}_3$	0.80	JB	105.32	101.50
	$\hat{c}_4$	-0.57	$\hat{c}_5$	-0.27	$\hat{c}_6$	-0.18	Kurt	3.37	3.36
	$\hat{c}_7$	-0.07	$\hat{c}_8$	0.01	$\hat{c}_9$	-0.03	Skew	-0.11	-0.11
Beijing	$\hat{c}_1$	3.89	$\hat{c}_2$	0.70	$\hat{c}_3$	0.84	JB	219.67	212.46
	$\hat{c}_4$	-0.22	$\hat{c}_5$	-0.49	$\hat{c}_6$	-0.20	Kurt	3.27	3.24
	$\hat{c}_7$	-0.14	$\hat{c}_8$	-0.11	$\hat{c}_9$	0.08	Skew	-0.28	-0.28
Taipei	$\hat{c}_1$	3.50	$\hat{c}_2$	1.49	$\hat{c}_3$	1.59	JB	181.90	169.41
	$\hat{c}_4$	-0.38	$\hat{c}_5$	-0.16	$\hat{c}_6$	0.03	Kurt	3.26	3.24
	$\hat{c}_7$	-0.17	$\hat{c}_8$	-0.09	$\hat{c}_9$	-0.18	Skew	-0.39	-0.37

Table 4.3: First 7 coefficients  $\{c_l\}_{l=1}^7$  of seasonal variance  $\sigma_t^2$  fitted with a Fourier truncated series. The coefficients are significant at 1% level. Skewness (Skew), kurtosis (Kurt) and values of Jarque Bera (JB) test statistics of standardized residuals with seasonal variances fitted with GARCH-Fourier series  $\hat{\varepsilon}_t/\hat{\sigma}_{t,FTSG}$  and with local linear regression  $\hat{\varepsilon}_t/\hat{\sigma}_{t,LLR}$ . Critical value at 5% significance level is 5.99, at 1% is -9.21.

#### 4 Weather Derivatives



Figure 4.9: Map of locations where temperature are collected

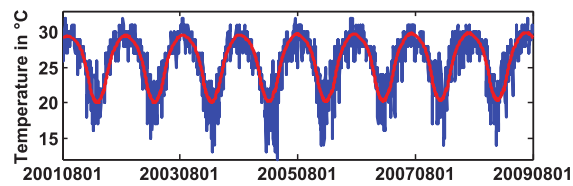


Figure 4.10: Daily average temperature (blue line) and fourier truncated seasonality function (red line) for Koahsiung.

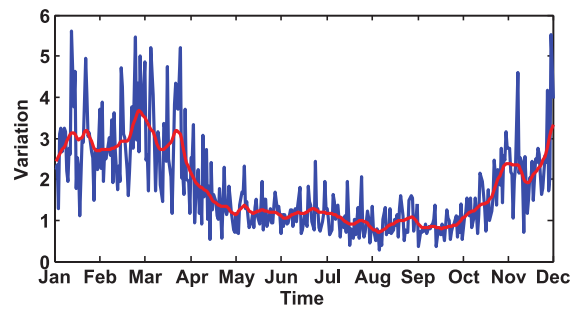


Figure 4.11: Empirical (blue line) and Local linear regression (red line) seasonal variation function for Koahsiung.

#### 4 Weather Derivatives

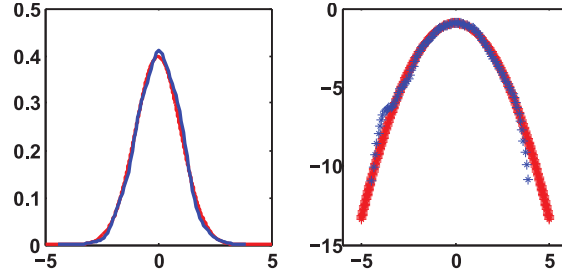


Figure 4.12: Kernel density estimates for standardized residuals ( $\frac{\hat{\varepsilon}_t}{\sigma_{t,LLR}}$ ) for Koashsiung (left panel) and Log densities normal fitting (solid line) and nonparametric fitting (dotted line) (right panel)

theoretical results provided in Diebold and Inoue [2001] or Granger and Hyung [2004]. We consider a model:

$$\begin{aligned} X_{t,j} &= T_{t,j} - \Lambda_t \\ X_{t,j} &= \sum_{l=1}^L \beta_l X_{t-l,j} + \sigma_t \varepsilon_{t,j} \\ \varepsilon_{t,j} &\sim \mathbf{N}(0, 1), i.i.d. \end{aligned} \quad (4.22)$$

with  $t = 1 \cdots I = 365$  days and  $j = 0 \cdots J$  years,  $\Lambda_t$  is the seasonality effect and  $\sigma_t$  is the seasonal variation. Note that (4.22) can be seen as a general setting of the model presented in Section 4.2.5. We are interested in modelling the seasonal volatility process  $\theta_t = \{\sigma_t^2\}$ .

Empirical evidence suggested that the parameters  $\Lambda_t, \sigma_t$  are  $t$  dependent, while the parameters  $\beta_l$ 's are more likely to be  $j$  dependent. For the estimation of  $\Lambda_t$ , we can either assume it is a deterministic and invariable component over years, estimated from a truncated Fourier series, or we estimated it iteratively by local smoothing technique. The  $\beta_l$ 's can be assumed to be estimated consistently as MLE from a global  $AR(L)$  model. For the simplicity of the setting, assume first to have  $X_{t,j}$  on hand. Then one needs to estimate parametrically  $\beta_j$ , and then estimate the volatility  $\sigma_t$  via a local adaptive procedure.

The temperature time serie  $T_{t,j}$  is approximated at a fixed time point  $s \in 1, 2, \dots, 365$ . Define a sequence of ordered weights  $W_s^{(k)} = (w(s, 1, h_k), w(s, 2, h_k), \dots, w(s, 365, h_k))^T$ , where  $w(s, t, h_k) = K_{h_k}(s - t)$ ,  $(h_1 < h_2 < \dots < h_K)$  as a sequence of bandwidth candidates with  $K(u) = 15(1 - u^2)^2 I(|u| \leq 1)/16$ . Our goal is to find out an optimal homogeneous interval, in which a global smoothing parameter could be applied. Consider the volatility estimation by maximizing the local Gaussian likelihood with weight

#### 4 Weather Derivatives

$w(s, t, h_k)$ , where  $k = 1, \dots, K$ :

$$\hat{\varepsilon}_{t,j} = X_{365j+t} - \sum_{l=1}^L \hat{\beta}_l X_{365j+t-l} \quad (4.23)$$

$$\begin{aligned} \tilde{\theta}^k(s) &= \arg \min_{\theta \in \Theta} \sum_{t=1}^{365} \sum_{j=0}^J \{ \log(2\pi\theta)/2 + \hat{\varepsilon}_{t,j}^2/2\theta \} w(s, t, h_k), \\ &\stackrel{\text{def}}{=} \arg \max_{\theta \in \Theta} -L(W_s^{(k)}, \theta) \end{aligned} \quad (4.24)$$

Then,

$$\begin{aligned} \tilde{\theta}^k(s) &= \sum_{t,j} \hat{\varepsilon}_{t,j}^2 w(s, t, h_k) / \sum_{t,j} w(s, t, h_k) \\ &= \sum_t \hat{\varepsilon}_t^2 w(s, t, h_k) / \sum_t w(s, t, h_k) \end{aligned}$$

with

$$\hat{\varepsilon}_t \stackrel{\text{def}}{=} (J+1)^{-1} \sum_{j=0}^J \hat{\varepsilon}_{t,j}^2.$$

with  $\hat{\varepsilon}_t \stackrel{\text{def}}{=} (J+1)^{-1} \sum_{j=0}^J \hat{\varepsilon}_{t,j}^2$ . For example, for Taipei  $J = 36$ , since the daily average temperature period is 19730101 – 20081230. From a smoother perspective we are in a comfortable situation since the boundary bias is not an issue, as we are dealing with a periodic function  $\theta_t : \theta_{t+365} = \theta_t$ . We invent the mirror observations. Assume  $h_K < 365/2$ , then the observations look like  $\hat{\varepsilon}_{-364}^2, \hat{\varepsilon}_{-363}^2, \dots, \hat{\varepsilon}_0^2, \hat{\varepsilon}_1^2, \dots, \hat{\varepsilon}_{730}^2$ , where

$$\begin{aligned} \hat{\varepsilon}_t^2 &\stackrel{\text{def}}{=} \hat{\varepsilon}_{365+t}^2, -364 \leq t \leq 0 \\ \hat{\varepsilon}_t^2 &\stackrel{\text{def}}{=} \hat{\varepsilon}_{t-365}^2, 366 \leq t \leq 730 \end{aligned}$$

The parametric Exponential Bound satisfies:

$$\begin{aligned} L(\tilde{\theta}, \theta^*) &\stackrel{\text{def}}{=} L(\tilde{\theta}) - L(\theta^*) = N\mathcal{K}(\tilde{\theta}, \theta^*) \\ \mathcal{K}(\theta, \theta^*) &= -\{\log(\theta/\theta^*) + 1 - \theta^*/\theta\}/2 \end{aligned}$$

where  $\mathcal{K}\{\theta, \theta^*\}$  is the Kullback Leibler divergence between  $\theta$  and  $\theta^*$  and  $N = 365 \times J$ . For any  $\mathfrak{z} > 0$ , with

$$\begin{aligned} \mathbb{P}_{\theta^*} \{L(\tilde{\theta}, \theta^*) > \mathfrak{z}\} &\leq 2 \exp(-\mathfrak{z}) \\ \mathbb{E}_{\theta^*} L(\tilde{\theta}, \theta^*)^r &\leq \mathfrak{r}_r, \end{aligned}$$

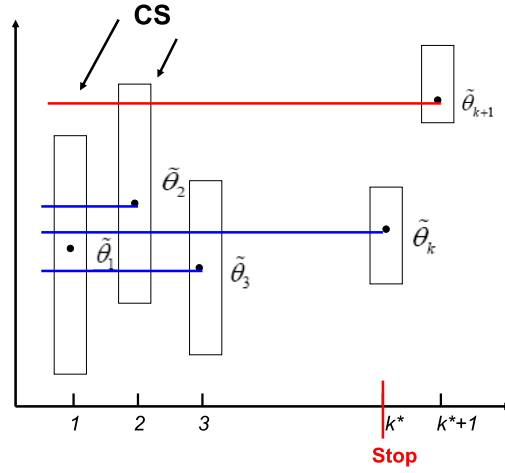


Figure 4.13: Localized model selection

where  $\mathfrak{r}_r = 2r \int_{\mathfrak{z} \geq 0} \mathfrak{z}^{r-1} \exp(-\mathfrak{z}) d\mathfrak{z}$ .

The aim of our estimation procedure is to build an estimate  $\hat{\theta}(s)$  which behaves as the best in the family  $\tilde{\theta}_k(s)$ , we drop  $s$  for the simplicity of our notation. The local adaptive procedure described in Spokoiny [2009] was adopted to our setting as follow:

- Fix a point  $s \in \{1, 2, \dots, 365\}$
- Start with the smallest interval parameter through  $h_1$ :

$$\hat{\theta}_s^1 = \tilde{\theta}_s^1$$

- For  $k \geq 2$ ,  $\tilde{\theta}_k$  is accepted and  $\hat{\theta}_k = \tilde{\theta}_k$  if  $\tilde{\theta}_{k-1}$  is accepted. Define the fitted likelihood ratio measuring the estimation risk:

$$LR(\tilde{\theta}_s^{(\ell)}, \tilde{\theta}_s^{(k)}) \stackrel{\text{def}}{=} L(W_s^{(\ell)}, \tilde{\theta}_s^{(\ell)}) - L(W_s^{(k)}, \tilde{\theta}_s^{(k)}) \leq \mathfrak{z}_\ell, \forall \ell < k - 1 \quad (4.25)$$

where  $\mathfrak{z}_\ell$  specify the critical values. Otherwise, stop the procedure and let  $\hat{\theta}_k = \hat{\theta}_{k-1}$ , where  $\hat{\theta}_k$  is the latest accepted after first  $k$  steps.

- Define  $\hat{k}$  as the  $k$ th step we stopped, and  $\hat{\theta}_\ell = \tilde{\theta}_{\hat{k}}, \ell \geq k$ .

The procedure is illustrated in Figure 4.13. For every estimate  $\tilde{\theta}_k$  the corresponding confidence interval is shown. If the horizontal line going through is center does not cross all the preeceding intervals then the procedure terminates.

A bound for the risk associated with first kind error is given by the propagation condition:

$$\mathbb{E}_{\theta^*} \frac{|L(W^{(k)}, \tilde{\theta}_k, \hat{\theta})|^r}{\mathfrak{r}_r} \leq \alpha \quad (4.26)$$

#### 4 Weather Derivatives

where  $k = 1, \dots, K$  and  $\tau_r$  is the parametric risk bound.

To run the algorithm, we need to specify the critical values. The sequential choice of critical values is described as below:

- Consider first  $\xi_1$  and let  $\xi_2 = \xi_3 = \dots = \xi_{K-1} = \infty$ . This leads to the estimates  $\hat{\theta}_k(\xi_1)$  and the value  $\xi_1$  is selected as the minimal one for which

$$\sup_{\theta^*} E_{\theta^*} \frac{|L\{W^{(k)}, \tilde{\theta}_k, \hat{\theta}_k(\xi_1)\}|^r}{\tau_r} \leq \frac{\alpha}{K-1}, k = 2, \dots, K.$$

where  $\alpha$  is the significant level of the test and  $\tau_r$  is the risk bound in the parametric model.

- Suppose  $\xi_1, \dots, \xi_{k-1}$  have been fixed, we set  $\xi_k = \dots = \xi_{K-1} = \infty$  and fix  $\xi_k$  leading to the set of parameter  $\xi_1, \dots, \xi_k, \infty, \dots, \infty$  and the estimates  $\hat{\theta}_m(\xi_1, \dots, \xi_k)$  for  $m = k+1, \dots, K$ . Then  $\xi_k$  is selected as the minimal value which fulfills

$$\sup_{\theta^*} E_{\theta^*} \frac{|L\{W^{(k)}, \tilde{\theta}_m, \hat{\theta}_m(\xi_1, \xi_2, \dots, \xi_k)\}|^r}{\tau_r} \leq \frac{k\alpha}{K-1},$$

for  $m = k+1, \dots, K$ .

A further refined approach is to consider an iterative procedure that could be applied to estimate the seasonal components  $\{\Lambda_t, \sigma_t\}$  better. The procedure is:

- Estimate the initial  $\Lambda_t^{(0)}$  using a truncated Fourier series or a local smoother.
- Fixed  $\hat{\Lambda}_t^{(i)}, i = 0, 1, 2, \dots$  from last step, and fix  $\hat{\beta}$ , get  $\hat{\theta}^{(i)}$  using the above mentioned local adaptive procedure
- Fixed  $\hat{\theta}^{(i)}$  and  $\hat{\beta}$ ,  $\hat{\Lambda}_t^{(i+1)}, i = 0, 1, 2, \dots$  is estimated via another local adaptive procedure:

$$\arg \min_{\{a,b\}^\top} \sum_{t=1}^{365} \sum_{j=0}^J \{Y_{365j+t} - \sum_{l=1}^L \hat{\beta}_l X_{t-l,j}\}^2 w(s, t, h'_k) / 2\hat{\theta}^{(i)} \quad (4.27)$$

where  $\{h'_1, h'_2, h'_3, \dots, h'_{K'}\}$  is a sequence of bandwidths.

- Fix  $\hat{\Lambda}_t^{(i)}$  and  $\hat{\beta}$  to get  $\hat{\theta}^{(i+1)}$  using the same procedure.
- Terminate the iteration till both  $|\hat{\Lambda}_t^{(i+1)} - \hat{\Lambda}_t^{(i)}| < \pi_1$  and  $|\hat{\theta}_t^{(i+1)} - \hat{\theta}_t^{(i)}| < \pi_2$ , where  $\pi_1$  and  $\pi_2$  determined values.

For more details of this adaptive method, see Haerdle et al. [2010] and Spokoiny [2009].

### 4.3 Stochastic Pricing model

As temperature is not a marketable asset, the replication arguments for any temperature futures do not hold and incompleteness of the market follows. Here the connection spot-future is not clear since the underlying spot is not restorable. Hence, the spot is not accounted of being a tradable asset, and will not be a part of a definition when fixing a martingale measure. In this context, any probability measure  $Q$  equivalent to the objective  $P$  is also an equivalent martingale measure and a risk neutral probability turns all the tradable assets into martingales after discounting. This implies that there is not need to restrict the temperature price model to the class of semimartingales. However, since temperature future/option prices dynamics are indeed tradable assets, they must be free of arbitrage, thus we require to have the semimartingale property in the dynamics of future prices to ensure the existence of risk neutral probabilities.

Thanks to the Girsanov theorem, equivalent changes of measures are simply associated with changes of drift. Hence, under a probability space  $(\Omega, \mathcal{F}, Q)$  with a filtration  $\{\mathcal{F}_t\}_{0 \leq t \leq \tau_{\max}}$ , where  $\tau_{\max}$  denotes a maximal time covering all times of interest in the market, we choose a parameterized equivalent pricing measure  $Q = Q_\theta$  that completes the market and pin it down to compute the arbitrage free temperature future price:

$$F_{(t, \tau_1, \tau_2)} = E^{Q_\theta} [Y | \mathcal{F}_t] \quad (4.28)$$

where  $Y$  refers to the payoff from the (CAT/HDD/CDD) temperature index. The future temperature price with delivery over a period  $[\tau_1, \tau_2]$  is expressed through the average (or weighted average) of the spot temperature over the delivery period.

The risk adjusted probability measure can be retrieved via Girsanov's theorem by establishing:

$$B_t^\theta = B_t - \int_0^t \theta_u du \quad (4.29)$$

where  $B_t^\theta$  is a Brownian motion for any time before the end of the trading time ( $t \leq \tau_{\max}$ ) and a martingale under  $Q_\theta$ .  $\theta$  denotes the market price of risk (MPR) and it is a real valued, bounded and piecewise continuous function. The market price of risk is nothing else than the return in excess of the risk-free rate that the market wants as compensation for taking risk. We later relax that assumption, by considering the time dependent market price of risk  $\theta_t$ . Note that the definition of  $\theta$  in this section is different from the  $\theta$  considered in section 4.2.6.

The non-arbitrage price of a temperature derivative requires a stochastic model for the temperature dynamics. We put into performance the existing pricing methodology of Benth et al. [2007b], which assumes a continuous-time process stochastic AR(p) (CAR(p)) for the temperature modelling. This methodology supposes a vectorial OU process  $\mathbf{X}_t \in \mathbb{R}^p$  for  $p \geq 1$ :

$$d\mathbf{X}_t = \mathbf{A}\mathbf{X}_t dt + \mathbf{e}_p \sigma_t dB_t \quad (4.30)$$



#### 4 Weather Derivatives

where  $\mathbf{e}_k$  denotes the  $k$ 'th unit vector in  $\mathbb{R}^p$  for  $k = 1, \dots, p$ ,  $\sigma_t > 0$  states the volatility,  $B_t$  is a Brownian motion and  $\mathbf{A}$  is a  $p \times p$ -matrix:

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & & \ddots & & 0 & \vdots \\ 0 & \dots & \dots & 0 & 0 & 1 \\ -\alpha_p & -\alpha_{p-1} & \dots & 0 & -\alpha_1 & \end{pmatrix} \quad (4.31)$$

with positive constants  $\alpha_k$ . Following this nomenclature, the state vector  $\mathbf{X}_t$  is namely the temperatures after removing seasonality at times  $t, t-1, t-2, t-3 \dots X_{q(t)}$  with  $q = 1, \dots, p$  is the  $q$ 'th coordinate of the  $\mathbf{X}_t$ . Setting  $q = 1$  is equivalent to the temperature time series  $T_t = \Lambda_t + X_{1(t)}$ . By applying the multidimensional Itô Formula, the process in (4.30) with  $\mathbf{X}_t = \mathbf{x} \in \mathbb{R}^p$  has the explicit form:

$$\mathbf{X}_s = \exp \{ \mathbf{A}(s-t) \} \mathbf{x} + \int_t^s \exp \{ \mathbf{A}(s-u) \} \mathbf{e}_p \sigma_u dB_u \quad (4.32)$$

for  $s \geq t \geq 0$  and stationarity holds when the eigenvalues of  $\mathbf{A}$  (4.31) have negative real part or the variance matrix  $\int_0^t \sigma_{t-s}^2 \exp \{ \mathbf{A}(s) \} \mathbf{e}_p \mathbf{e}_p^\top \exp \{ \mathbf{A}^\top(s) \} ds$  converges as  $t \rightarrow \infty$ . An  $AR(1)$  may be a reasonable approximation and will correspond to an OU process. The  $AR(3)$  estimated in Section 4.2.5 can be therefore seen as a discretely sampled continuous-time processes ( $CAR(p)$ ) (4.30) driven by a one dimensional Brownian motion, though the continuous time process is Markov in higher dimension. We now derive an analytical link between  $X_{1(t)}, X_{2(t)}$  and  $X_{3(t)}$  and the lagged temperatures up to time  $t-3$ . We approximate by Euler discretization to get  $X_{1(t+3)}$ . Thus for:  $p = 1, \mathbf{X}_t = X_{1(t)}$  and (4.30) becomes  $dX_{1(t)} = -\alpha_1 X_{1(t)} dt + \sigma_t dB_t$ , which is the continuous version of an  $AR(1)$  process. Similarly, for  $p = 2$ , assuming a time step of length one  $dt = 1$ , substitute  $X_{2(t)}$  iteratively to get  $X_{1(t+2)} \approx (2 - \alpha_1)X_{1(t+1)} + (\alpha_1 - \alpha_2 - 1)X_{1(t)} + \sigma_t \varepsilon_t$ , where  $\varepsilon_t = B_{t+1} - B_t$ . For  $p = 3$ , we have:

$$\begin{aligned} X_{1(t+1)} - X_{1(t)} &= X_{2(t)} dt \\ X_{2(t+1)} - X_{2(t)} &= X_{3(t)} dt \\ X_{3(t+1)} - X_{3(t)} &= -\alpha_3 X_{1(t)} dt - \alpha_2 X_{2(t)} dt - \alpha_1 X_{3(t)} dt + \sigma_t \varepsilon_t \\ &\dots \\ X_{1(t+3)} - X_{1(t+2)} &= X_{2(t+2)} dt \\ X_{2(t+3)} - X_{2(t+2)} &= X_{3(t+2)} dt \\ X_{3(t+3)} - X_{3(t+2)} &= -\alpha_3 X_{1(t+2)} dt - \alpha_2 X_{2(t+2)} dt - \alpha_1 X_{3(t+2)} dt + \sigma_t \varepsilon_t \end{aligned} \quad (4.33)$$

substituting into the  $X_{1(t+3)}$  dynamics and setting  $dt = 1$ :

$$X_{1(t+3)} \approx (3 - \alpha_1)X_{1(t+2)} + (2\alpha_1 - \alpha_2 - 3)X_{1(t+1)} + (-\alpha_1 + \alpha_2 - \alpha_3 + 1)X_{1(t)} \quad (4.34)$$

#### 4 Weather Derivatives

Please note that this corrects the derivation in Benth et al. [2007b]. The approximation (4.34) is required to compute the eigenvalues of matrix  $\mathbf{A}$ . Table (4.2) displays the  $CAR(3)$ -parameters, the eigenvalues of the matrix  $\mathbf{A}$  for the studied datas. The stationarity condition is fulfilled since the eigenvalues of  $\mathbf{A}$  have negative real parts.

Since dynamics of temperature future prices must be free of arbitrage, under the pricing equivalent measure  $Q_\theta$ , the temperature dynamics of (4.30) become

$$d\mathbf{X}_t = (\mathbf{A}\mathbf{X}_t + \mathbf{e}_p\sigma_t\theta_t)dt + \mathbf{e}_p\sigma_t dB_t^\theta \quad (4.35)$$

with explicit dynamics, for  $s \geq t \geq 0$ :

$$\mathbf{X}_s = \exp\{\mathbf{A}(s-t)\}\mathbf{x} + \int_t^s \exp\{\mathbf{A}(s-u)\}\mathbf{e}_p\sigma_u\theta_u du + \int_t^s \exp\{\mathbf{A}(s-u)\}\mathbf{e}_p\sigma_u dB_u^\theta$$

By inserting (4.1), (4.2), (4.3) into (4.28), Benth et al. [2007b] calculate explicitly the risk neutral prices for HDD, CDD and CAT futures for contracts traded before the temperature measurement period, i.e.  $0 \leq t \leq \tau_1 < \tau_2$ :

$$\begin{aligned} F_{HDD}(t, \tau_1, \tau_2) &= \int_{\tau_1}^{\tau_2} v_{t,s} \psi \left[ \frac{c - m_{\{t,s, \mathbf{e}_1^\top \exp\{\mathbf{A}(s-t)\}\mathbf{X}_t\}}}{v_{t,s}} \right] ds \\ F_{CDD}(t, \tau_1, \tau_2) &= \int_{\tau_1}^{\tau_2} v_{t,s} \psi \left[ \frac{m_{\{t,s, \mathbf{e}_1^\top \exp\{\mathbf{A}(s-t)\}\mathbf{X}_t\}} - c}{v_{t,s}} \right] ds \\ F_{CAT}(t, \tau_1, \tau_2) &= \int_{\tau_1}^{\tau_2} \Lambda_u du + \mathbf{a}_{t, \tau_1, \tau_2} \mathbf{X}_t + \int_t^{\tau_1} \theta_u \sigma_u \mathbf{a}_{t, \tau_1, \tau_2} \mathbf{e}_p du \\ &\quad + \int_{\tau_1}^{\tau_2} \theta_u \sigma_u \mathbf{e}_1^\top \mathbf{A}^{-1} [\exp\{\mathbf{A}(\tau_2 - u)\} - I_p] \mathbf{e}_p du \end{aligned} \quad (4.36)$$

with  $\mathbf{a}_{t, \tau_1, \tau_2} = \mathbf{e}_1^\top \mathbf{A}^{-1} [\exp\{\mathbf{A}(\tau_2 - t)\} - \exp\{\mathbf{A}(\tau_1 - t)\}]$ ,  $I_p$ - $p \times p$  identity matrix,

$$\begin{aligned} m_{\{t,s,x\}} &= \Lambda_s + \int_t^s \sigma_u \theta_u \mathbf{e}_1^\top \exp\{\mathbf{A}(s-t)\} \mathbf{e}_p du + x \\ v_{t,s}^2 &= \int_t^s \sigma_u^2 \left[ \mathbf{e}_1^\top \exp\{\mathbf{A}(s-t)\} \mathbf{e}_p \right]^2 du \end{aligned} \quad (4.37)$$

and  $\psi(x) = x\Phi(x) + \varphi(x)$  with  $x = \mathbf{e}_1^\top \exp\{\mathbf{A}(s-t)\} \mathbf{X}_t$ .

The future prices are given by the accumulated mean temperature and market price of risk over the measurement period, plus an effect of the autoregressive process of temperature.

The explicit formulae for the CAT call option written on a CAT future with strike  $K$

#### 4 Weather Derivatives

at exercise time  $\tau < \tau_1$  during the period  $[\tau_1, \tau_2]$  is:

$$C_{CAT}(t, \tau, \tau_1, \tau_2) = \exp \{-r(\tau - t)\} \times \left[ \left( F_{CAT}(t, \tau_1, \tau_2) - K \right) \Phi \{d(t, \tau, \tau_1, \tau_2)\} + \int_t^\tau \Sigma_{CAT}(s, \tau_1, \tau_2) ds \phi \{d(t, \tau, \tau_1, \tau_2)\} \right] \quad (4.38)$$

where  $d(t, \tau, \tau_1, \tau_2) = \frac{F_{CAT}(t, \tau_1, \tau_2) - K}{\sqrt{\int_t^\tau \Sigma_{CAT}(s, \tau_1, \tau_2) ds}}$  and  $\Sigma_{CAT}(s, \tau_1, \tau_2) = \sigma_t \mathbf{a}_{t, \tau_1, \tau_2} \mathbf{e}_p$  and  $\Phi$  denotes the standard normal cdf. The option can be perfectly hedged once the specification of the risk neutral probability  $Q^\theta$  determines the complete market of futures and options. Then, the option price will be the unique cost of replication.

To replicate the call option with CAT-futures, one should compute the number of CAT-futures held in the portfolio, which is simply computed by the option's delta:

$$\Phi \{d(t, T, \tau_1, \tau_2)\} = \frac{\partial C_{CAT}(t, \tau, \tau_1, \tau_2)}{\partial F_{CAT}(t, \tau_1, \tau_2)} \quad (4.39)$$

The strategy holds close to zero CAT futures when the option is far out of the money, close to 1 otherwise.

In the lower left hand side of Figure 4.14 we plot the Berlin CAT volatility path  $\sigma_t \mathbf{a}_{t, \tau_1, \tau_2} \mathbf{e}_p$  for contracts issued before and within the measurement periods 2004-2008 and in the lower right hand side we display the plot for 2006. For contracts traded within the measurement period CAT volatility is close to zero when the time to measurement is large and it decreases up to the end of the measurement period. While, for contracts traded before the measurement period CAT volatility is also close to zero when the time to measurement is large, but increases up to the start of the measurement period.

In Figure 4.15 two Berlin CAT contracts issued on 20060517 are plotted: one with measurement period the first week of June and the other one as the whole month of June. The contract with the longest measurement period has the largest volatility. In contrast to the later effect, one can observe the effect of the  $CAR(3)$  in both contracts when the volatility decays just before maturity of the contracts. These two effects observed on Berlin CAT futures are also similar for Stockholm CAT futures, Benth et al. [2007b], however the deviations are less smoothed for Berlin.

For the call option written CDD-future, Benth et al. [2007b] found no analytical solution, but an expression suitable for Monte Carlo simulation. The risk neutral price of a CDD call written on a CDD future with strike  $K$  at exercise time  $\tau < \tau_1$  during the period  $[\tau_1, \tau_2]$ :

$$C_{CDD}(t, T, \tau_1, \tau_2) = \exp \{-r(\tau - t)\} \times E^{Q_\theta} \left[ \max \left( \int_{\tau_1}^{\tau_2} v_{\tau, s} \psi \left( \frac{m_{\text{index}} - c}{v_{t, s}} \right) ds - K, 0 \right) \right]_{\mathbf{x}=\mathbf{x}_t} \quad (4.40)$$

#### 4 Weather Derivatives

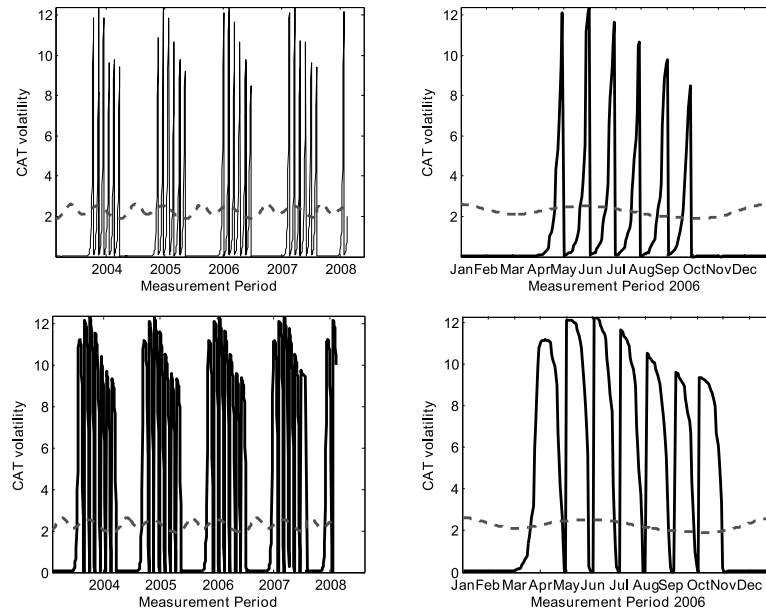


Figure 4.14: The Berlin CAT term structure of volatility (black line) and  $\sigma_t$  (dash line) from 2004-2008 (left) and 2006 (right) for contracts traded before (upper panel) and within (lower panel) the measurement period.

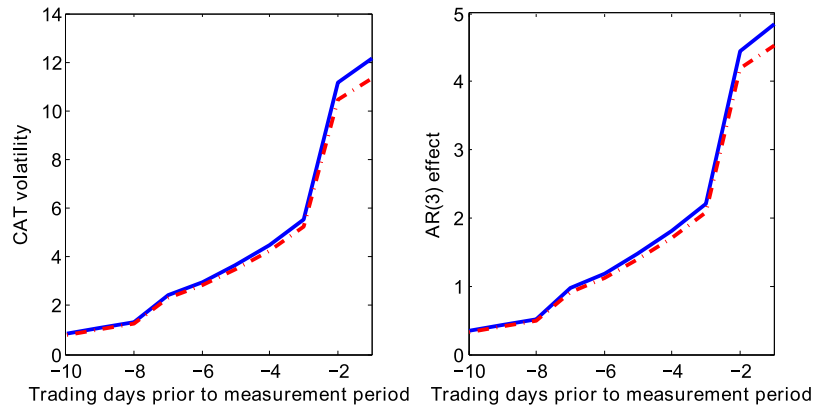


Figure 4.15: Berlin CAT volatility and AR(3) effect of 2 contracts issued on 20060517: one with whole June as measurement period (blue line) and the other one with only the 1st week of June (red line)

#### 4 Weather Derivatives

where  $\text{index} = \tau, s, \mathbf{e}_1^\top \exp \{ \mathbf{A}(s-t) \} \mathbf{x} + \int_t^\tau \mathbf{e}_1^\top \exp \{ \mathbf{A}(s-u) \} \mathbf{e}_p \sigma_u \theta_u du + \Sigma_{s,t,\tau} Y$ ,  $Y \sim \mathbf{N}(0, 1)$  and  $\Sigma_{s,t,\tau}^2 = \int_t^\tau [\mathbf{e}_1^\top \exp \{ \mathbf{A}(s-u) \} \mathbf{e}_p]^2 \sigma_u^2 du$ .

Since there is not analytical form of the CDD price, the hedging strategy cannot be inferred by differentiation. However, by the Clark-Ocone Formula (Karatzas et al. [1991]) applied to arbitrage pricing, the payoff of the call  $C = \max \{ F_{CDD}(t, T, \tau_1, \tau_2) - K, 0 \}$  can be represented as:

$$C = E^{Q_\theta} [C] + \int_0^\tau E^{Q_\theta} [D_t C | \mathcal{F}_t] dB_t^\theta \quad (4.41)$$

where  $D_t$  is the Malliavin derivative of the random variable  $C$ , see Malliavin and Thalmaier [2006]. Supposing that the volatility

$$\begin{aligned} \Sigma_{CDD}(t, T, \tau_1, \tau_2) &= \sigma_t \int_{\tau_1}^{\tau_2} \mathbf{e}_1^\top \exp \{ \mathbf{A}(s-t) \} \mathbf{e}_p \\ &\quad \times \psi \left( \frac{m_{\tau,s} \mathbf{e}_1^\top \exp \{ \mathbf{A}(s-t) \} \mathbf{x}_t}{v_{\tau,s}} \right) ds \end{aligned} \quad (4.42)$$

is positive, then (4.41) is equal to:

$$C = E^{Q_\theta} [C] + \int_0^\tau \Sigma_{CDD}^{-1}(t, T, \tau_1, \tau_2) \tilde{\zeta}(t, \tau) dF_{CDD}(t, T, \tau_1, \tau_2) \quad (4.43)$$

where

$$\begin{aligned} \tilde{\zeta}(t, \tau) &= \sigma_t E^{Q_\theta} \left[ \mathbf{1} \left\{ \int_{\tau_1}^{\tau_2} v_{\tau,s} \psi \left( \frac{m(\tau,s,Z)}{v_{\tau,s}} \right) ds > K \right\} \right. \\ &\quad \left. \times \int_{\tau_1}^{\tau_2} \mathbf{e}_1^\top \exp \{ \mathbf{A}(s-t) \} \mathbf{e}_p \Phi \left( \frac{m(\tau,s,Z)}{v_{\tau,s}} \right) ds \right]_{\mathbf{x}=\mathbf{x}_t} \end{aligned} \quad (4.44)$$

and  $m_{(\tau,s,Z)}$  defined as in (4.37) and  $Z$  is normally distributed with mean

$$\mathbf{e}_1^\top \exp \{ \mathbf{A}(s-t) \} \mathbf{x} + \int_t^\tau \mathbf{e}_1^\top \exp \{ \mathbf{A}(s-u) \} \mathbf{e}_p \sigma_u \theta_u du$$

and variance

$$\int_t^\tau [\mathbf{e}_1^\top \exp \{ \mathbf{A}(s-u) \} \mathbf{e}_p]^2 \sigma_u^2 du$$

Thus, the hedging strategy is explicitly given as:

$$h_{CDD}(t, T, \tau_1, \tau_2) = \Sigma_{CDD}^{-1}(t, T, \tau_1, \tau_2) \tilde{\zeta}(t, \tau)$$

#### 4.4 The implied market price of weather risk

For pricing and hedging non-tradable assets one essentially needs to incorporate the market price of risk (MPR), which is an important parameter of the associated equivalent martingale measure. The MPR measures the additional return for bearing more risk and it adjusts the underlying process so that the level of the risk aversion is not needed for valuation. This section deals exactly with the differences between historical and risk neutral behaviors of temperature.

Using statistical modelling and given that liquid derivatives contracts based on daily temperature are traded on the Chicago Mercantile Exchange (CME), the MPR (the change of drift) is implied from traded futures type contracts (CAT, CDD, HDD and AAT) for different cities. Similarly, one can infer the MPR from option data and compare the findings with prices in the future market. The concept of implied MPR is similar to that used in extracting implied volatilities, Fengler et al. [2007]. Once we know the MPR for temperature futures, then we know the MPR for options and thus one can price new non-standard maturities derivatives.

This study is to be seen between a calibration procedure for financial engineering purposes and an economic and statistical testing approach. In the former case, a single date (but different time horizons and calibrated instruments are used) is required, since the model is recalibrated daily to detect intertemporal effects. In the latter case, we start from a specification of the MPR and check consistency with the data. Different specifications of the MPR are investigated: a constant, a piece-wise linear function, a 2 piece-wise linear function, a time-deterministic function and a financial-bootstrapping MPR. Since smoothing individual estimates is fundamentally different from estimating a deterministic function, the results are also assured by fitting a parametric function to all available contract prices (calendar year estimation). In particular, the study concentrates on monthly contracts, but similar implications can be done for seasonal strip contracts. The weather future data periods are: Portland and Atlanta (20000711-20091113), New York (20000605-20091113), Houston (20030925-20091113), Berlin (20031006-20091113), Essen (20050617-20091113), Tokyo and Osaka (20040723-20091113).

##### 4.4.1 Constant market price of risk for different daily contract

Given observed CAT/HDD/CDD future prices traded at CME and by inverting (4.36), the MPR  $\theta$  is implied for a single date  $t$  for contracts traded at different time horizons  $i = 1 \cdots I$  with monthly measurement period  $[\tau_1^i, \tau_2^i]$  and  $t \leq \tau_1^i < \tau_2^i$ . Our first assump-

## 4 Weather Derivatives

tion is to set  $\theta_t$  as a constant over  $[\tau_1^i, \tau_2^i]$ :

$$\begin{aligned} \arg \min_{\hat{\theta}_{t,CAT}^i} & \left( F_{CAT}(t, \tau_1^i, \tau_2^i) - \int_{\tau_1^i}^{\tau_2^i} \hat{\Lambda}_u du - \hat{\mathbf{a}}_{t, \tau_1^i, \tau_2^i} \mathbf{X}_t - \hat{\theta}_{t,CAT}^i \left\{ \int_t^{\tau_1^i} \hat{\sigma}_u \hat{\mathbf{a}}_{t, \tau_1^i, \tau_2^i} \mathbf{e}_p du \right. \right. \\ & \left. \left. + \int_{\tau_1^i}^{\tau_2^i} \hat{\sigma}_u \mathbf{e}_1^\top \mathbf{A}^{-1} \left[ \exp \left\{ \mathbf{A}(\tau_2^i - u) \right\} - I_p \right] \mathbf{e}_p du \right\} \right)^2 \\ \arg \min_{\hat{\theta}_{t,HDD}^i} & \left( F_{HDD}(t, \tau_1^i, \tau_2^i) - \int_{\tau_1^i}^{\tau_2^i} v_{t,s} \psi \left[ \frac{c - \hat{m}_{\{t,s, \mathbf{e}_1^\top \exp \{ \mathbf{A}(s-t) \} \mathbf{X}_t \}}^1}{v_{t,s}} \right] ds \right)^2 \end{aligned} \quad (4.45)$$

with  $\hat{m}_{\{t,s,x\}}^1 = \Lambda_s + \hat{\theta}_{t,HDD}^i \int_t^s \sigma_u \mathbf{e}_1^\top \exp \{ \mathbf{A}(s-t) \} \mathbf{e}_p du + x$ , and  $v_{t,s}^2, \psi(x)$  and  $x$  defined as in (4.37). The case of the MPR for CDD futures  $\hat{\theta}_{t,CDD}^i$  is similar to the HDD case (4.45). In the next sections, we will therefore omit CDD parametrizations. Table 4.4 and the first and second panel of Figure 4.17 show the CAT-Berlin, CAT-Essen and AAT-Tokyo future prices and constant MPR estimates.

### 4.4.2 Constant market price of risk per trading day

A simpler parametrization of  $\theta$  is to assume that it is constant across all time horizons contracts priced in a particular date. We therefore estimate this constant  $\theta_t$  (OLS MPR) for all CAT/HDD/CDD contracts with  $t \leq \tau_1^i < \tau_2^i, i = 1, \dots, I$ :

$$\begin{aligned} \arg \min_{\hat{\theta}_{t,CAT}} \sum_{i=1}^I & \left( F_{CAT}(t, \tau_1^i, \tau_2^i) - \int_{\tau_1^i}^{\tau_2^i} \hat{\Lambda}_u du - \hat{\mathbf{a}}_{t, \tau_1^i, \tau_2^i} \mathbf{X}_t - \hat{\theta}_{t,CAT} \left\{ \int_t^{\tau_1^i} \hat{\sigma}_u \hat{\mathbf{a}}_{t, \tau_1^i, \tau_2^i} \mathbf{e}_p du \right. \right. \\ & \left. \left. + \int_{\tau_1^i}^{\tau_2^i} \hat{\sigma}_u \mathbf{e}_1^\top \mathbf{A}^{-1} \left[ \exp \left\{ \mathbf{A}(\tau_2^i - u) \right\} - I_p \right] \mathbf{e}_p du \right\} \right)^2 \\ \arg \min_{\hat{\theta}_{t,HDD}} \sum_{i=1}^I & \left( F_{HDD}(t, \tau_1^i, \tau_2^i) - \int_{\tau_1^i}^{\tau_2^i} v_{t,s} \psi \left[ \frac{c - \hat{m}_{\{t,s, \mathbf{e}_1^\top \exp \{ \mathbf{A}(s-t) \} \mathbf{X}_t \}}^2}{v_{t,s}} \right] ds \right)^2 \end{aligned} \quad (4.46)$$

with  $\hat{m}_{\{t,s,x\}}^2 = \Lambda_s + \hat{\theta}_{t,HDD} \int_t^s \sigma_u \mathbf{e}_1^\top \exp \{ \mathbf{A}(s-t) \} \mathbf{e}_p du + x$ , and  $v_{t,s}^2, \psi(x), x$  defined in (4.36). The empirical results of this MPR parametrization based on CAT-Berlin, CAT-Essen and AAT-Tokyo futures are summarized in Table 4.4, where we observed that this is a very robust least squares estimation.

### 4.4.3 Two constant market prices of risk per trading day

Assuming now that, instead of one constant market price of risk per trading day, we have a step function with jump  $\hat{\theta}_t = I(u \leq \xi) \hat{\theta}_t^1 + I(u > \xi) \hat{\theta}_t^2$  with a given jump point  $\xi$  (take for example the first 150 days before the beginning of the measurement period).

#### 4 Weather Derivatives

Then we estimate  $\hat{\theta}_t$  with  $t \leq \tau_1^i < \tau_2^i$  (OLS2 MPR) for CAT/HDD/CDD contracts by:

$$\begin{aligned}
 f(\xi) = \arg \min_{\hat{\theta}_{t,CAT}^1, \hat{\theta}_{t,CAT}^2} \sum_{i=1}^I & \left( F_{CAT}(t, \tau_1^i, \tau_2^i) - \int_{\tau_1^i}^{\tau_2^i} \hat{\Lambda}_u du - \hat{\mathbf{a}}_{t, \tau_1^i, \tau_2^i} \mathbf{X}_t \right. \\
 & - \hat{\theta}_{t,CAT}^1 \left\{ \int_t^{\tau_1^i} I(u \leq \xi) \hat{\sigma}_u \hat{\mathbf{a}}_{t, \tau_1^i, \tau_2^i} \mathbf{e}_p du \right. \\
 & + \left. \int_{\tau_1^i}^{\tau_2^i} I(u \leq \xi) \hat{\sigma}_u \mathbf{e}_1^\top \mathbf{A}^{-1} \left[ \exp \left\{ \mathbf{A}(\tau_2^i - u) \right\} - I_p \right] \mathbf{e}_p du \right\} \\
 & - \hat{\theta}_{t,CAT}^2 \left\{ \int_t^{\tau_1^i} I(u > \xi) \hat{\sigma}_u \hat{\mathbf{a}}_{t, \tau_1^i, \tau_2^i} \mathbf{e}_p du \right. \\
 & + \left. \left. \int_{\tau_1^i}^{\tau_2^i} I(u > \xi) \hat{\sigma}_u \mathbf{e}_1^\top \mathbf{A}^{-1} \left[ \exp \left\{ \mathbf{A}(\tau_2^i - u) \right\} - I_p \right] \mathbf{e}_p du \right\} \right)^2 \quad (4.47)
 \end{aligned}$$

$$\begin{aligned}
 \arg \min_{\hat{\theta}_{t,HDD}^1, \hat{\theta}_{t,HDD}^2} \sum_{i=1}^I & \left( F_{HDD}(t, \tau_1^i, \tau_2^i) - \int_{\tau_1^i}^{\tau_2^i} v_{t,s} \psi \left[ \frac{c - \hat{m}_{\{t,s, \mathbf{e}_1^\top \exp \{ \mathbf{A}(s-t) \} \mathbf{X}_t\}}^3}{v_{t,s}} \right] ds \right)^2 \\
 \hat{m}_{\{t,s,x\}}^3 &= \Lambda_s + \hat{\theta}_{t,HDD}^1 \left\{ \int_t^s I(u \leq \xi) \sigma_u \mathbf{e}_1^\top \exp \{ \mathbf{A}(s-t) \} \mathbf{e}_p du + x \right\} \\
 & + \hat{\theta}_{t,HDD}^2 \left\{ \int_t^s I(u > \xi) \sigma_u \mathbf{e}_1^\top \exp \{ \mathbf{A}(s-t) \} \mathbf{e}_p du + x \right\} \quad (4.48)
 \end{aligned}$$

and  $v_{t,s}^2, \psi(x), x$  defined in (4.37). Table 4.4 and the third panel of Figure 4.17 show the CAT-Berlin, CAT-Essen and AAT-Tokyo - OLS2 MPR estimates of futures traded during (20031006-20090630). In the next step we optimized the value of  $\xi$  such as  $f(\xi)$  in (4.1) is minimized. Figure 4.16 shows the MPR estimates for Berlin CAT Future Prices traded on 20060530 with  $\xi = 62, 93, 123, 154$ . The line displays a discontinuity indicating that trading was not taking place. For  $\xi = 62, 93, 123, 154$  days, the corresponding sum of squared errors are 2759, 14794, 15191 and 15526. When the jump  $\xi$  is getting far from the measurement period, the value of the MPR  $\hat{\theta}_t^1$  decreases and  $\hat{\theta}_t^2$  increases, yielding a  $\hat{\theta}_t$  around zero.

##### 4.4.4 General form of the market price of risk per trading day

Generalising the piecewise continuous function given in the previous subsection, the (inverse) problem of determining  $\theta_t$  with  $t \leq \tau_1^i < \tau_2^i, i = 1, \dots, I$  can be formulated via



## 4 Weather Derivatives

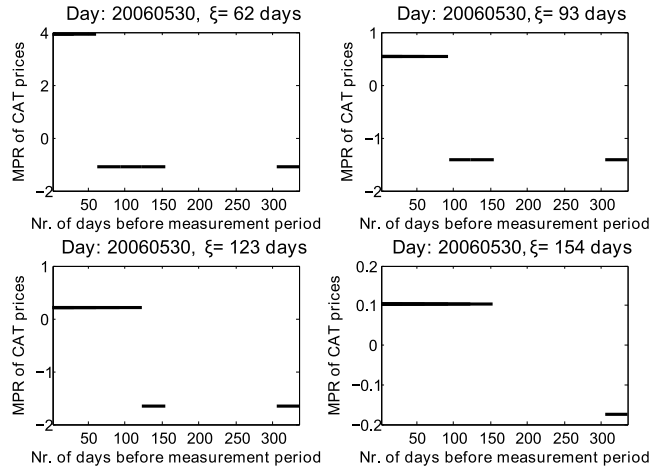


Figure 4.16: Two constant MPRs with  $\xi = 62, 93, 123, 154$  days (upper left, upper right, lower left, lower right) for Berlin CAT contracts traded on 20060530.

a series expansion for  $\theta_t$ :

$$\begin{aligned} \arg \min_{\hat{\gamma}_k} \sum_{i=1}^I & \left( F_{CAT}(t, \tau_1^i, \tau_2^i) - \int_{\tau_1^i}^{\tau_2^i} \hat{\Lambda}_u du - \hat{\mathbf{a}}_{t, \tau_1^i, \tau_2^i} \hat{\mathbf{X}}_t - \int_t^{\tau_1^i} \sum_{k=1}^K \hat{\gamma}_k \hat{h}_k(u_i) \hat{\sigma}_{u_i} \hat{\mathbf{a}}_{t, \tau_1^i, \tau_2^i} \mathbf{e}_p du_i \right. \\ & \left. - \int_{\tau_1^i}^{\tau_2^i} \sum_{k=1}^K \hat{\gamma}_k \hat{h}_k(u_i) \hat{\sigma}_{u_i} \mathbf{e}_1^\top \mathbf{A}^{-1} \left[ \exp \left\{ \mathbf{A}(\tau_2^i - u_i) \right\} - I_p \right] \mathbf{e}_p du_i \right)^2 \\ \arg \min_{\hat{a}_k} \sum_{i=1}^I & \left( F_{HDD}(t, \tau_1^i, \tau_2^i) - \int_{\tau_1^i}^{\tau_2^i} v_{t,s} \psi \left[ \frac{c - \hat{m}_{\{t,s, \mathbf{e}_1^\top \exp \{ \mathbf{A}(s-t) \} \mathbf{X}_t\}}^4}{v_{t,s}} \right] ds \right)^2 \end{aligned} \quad (4.49)$$

with  $\hat{m}_{\{t,s,x\}}^4 = \Lambda_s + \int_t^s \sum_{k=1}^K \hat{a}_k \hat{h}_k(u_i) \hat{\sigma}_{u_i} \mathbf{e}_1^\top \exp \{ \mathbf{A}(s-t) \} \mathbf{e}_p du_i + x$ , and  $v_{t,s}^2, \psi(x), x$  as (4.37).  $h_k(u_i), a_k(u_i)$  are vectors of known basis functions and may denote a B-spline basis for example.  $\gamma_k, \hat{h}_k$  define the coefficients and  $K$  is the number of knots. Table 4.4 and the fourth panel of Figure 4.17 display the spline MPR for CAT-Berlin, CAT-Essen and AAT-Tokyo Future Prices traded on 20060530 using cubic polynomials with  $k$  equal to the number of traded contracts  $i$ . This time dependent parametrization of the MPR (spline MPR) is close to zero, but it explodes for the days when trading is not taking place.

### 4.4.5 Bootstrapping the market price of risk

In this section, we propose a bootstrapping technique to get estimates of the MPR for CAT/HDD/CDD future contracts traded at time  $t \leq \tau_1^i < \tau_2^i$ , with  $\tau_1^i < \tau_1^{i+1} \leq \tau_2^i < \tau_2^{i+1}$  and  $i = 1 \dots I$ . This financial bootstrap idea consists of estimating the MPR  $\theta_t^i$  of

## 4 Weather Derivatives

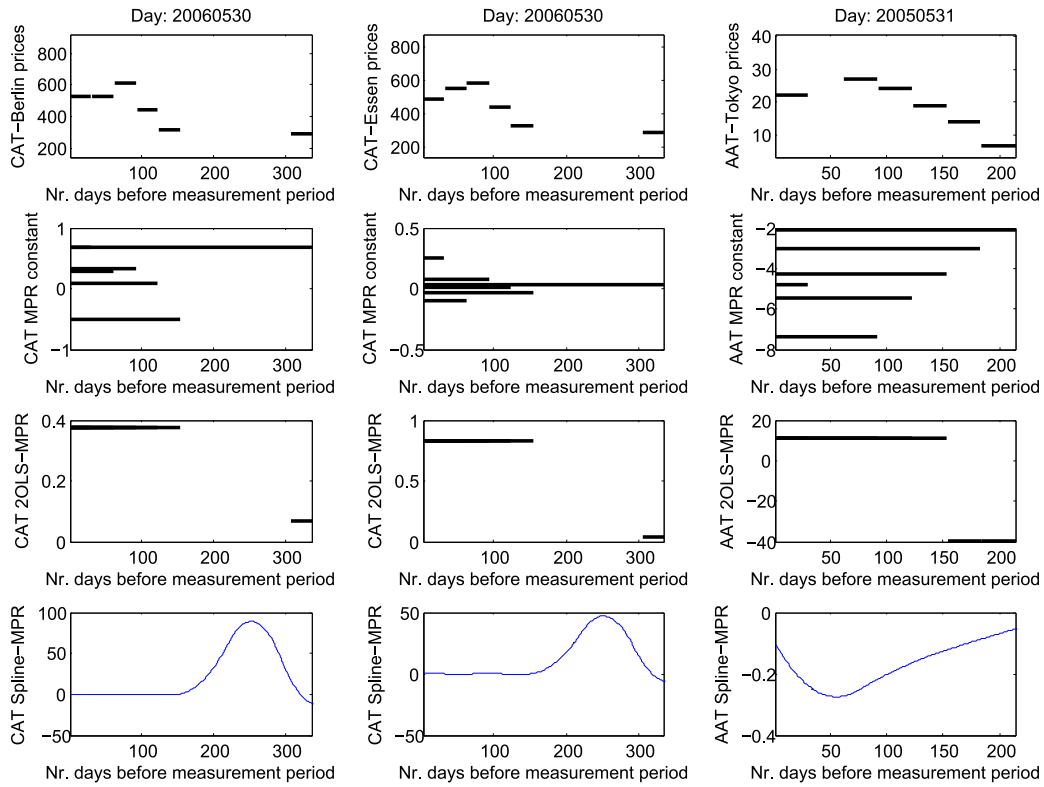


Figure 4.17: Prices (1 panel) and MPR for CAT-Berlin (left side), CAT-Essen (middle side), AAT-Tokyo (right side) of futures traded on 20050530 and 20060531. Constant MPR across contracts per trading day (2 panel), 2 constant per trading day OLS2-MPR (3 panel), time dependent MPR using spline (4 panel).

#### 4 Weather Derivatives

the future price contract with the closest measurement period by forward substitution and placing it into the estimation for the next MPR  $\theta_t^{i+1}$ . We do the estimation for CAT contracts, but the idea is the same also for HDD/CDD. Firstly,  $\hat{\theta}_{t,CAT}^1$  is estimated from the contract with  $i = 1$ :

$$\arg \min_{\hat{\theta}_{t,CAT}^1} \left( F_{CAT}(t, \tau_1^1, \tau_2^1) - \int_{\tau_1^1}^{\tau_2^1} \hat{\Lambda}_u du - \hat{\mathbf{a}}_{t, \tau_1^1, \tau_2^1} \hat{\mathbf{X}}_t - \hat{\theta}_{t,CAT}^1 \left\{ \int_t^{\tau_1^1} \hat{\sigma}_u \hat{\mathbf{a}}_{t, \tau_1^1, \tau_2^1} \mathbf{e}_p du + \int_{\tau_1^1}^{\tau_2^1} \hat{\sigma}_u \mathbf{e}_1^\top \mathbf{A}^{-1} [\exp \{ \mathbf{A}(\tau_2^1 - u) \} - I_p] \mathbf{e}_p du \right\} \right)^2 \quad (4.50)$$

Secondly,  $\hat{\theta}_{t,CAT}^1$  is substituted in the period  $(\tau_1^1, \tau_2^1)$  to get an estimate of  $\hat{\theta}_{t,CAT}^2$ :

$$\arg \min_{\hat{\theta}_{t,CAT}^2} \left( F_{CAT}(t, \tau_1^2, \tau_2^2) - \int_{\tau_1^2}^{\tau_2^2} \hat{\Lambda}_u du - \hat{\mathbf{a}}_{t, \tau_1^2, \tau_2^2} \hat{\mathbf{X}}_t - \int_t^{\tau_1^1} \hat{\theta}_{t,CAT}^1 \hat{\sigma}_u \hat{\mathbf{a}}_{t, \tau_1^2, \tau_2^2} \mathbf{e}_p du - \int_{\tau_1^2}^{\tau_2^2} \hat{\theta}_{t,CAT}^2 \hat{\sigma}_u \mathbf{e}_1^\top \mathbf{A}^{-1} [\exp \{ \mathbf{A}(\tau_2^2 - u) \} - I_p] \mathbf{e}_p du \right)^2 \quad (4.51)$$

Then substitute  $\hat{\theta}_{t,CAT}^1$  in the period  $(\tau_1^1, \tau_2^1)$  and  $\hat{\theta}_{t,CAT}^2$  in the period  $(\tau_1^2, \tau_2^2)$  to estimate  $\hat{\theta}_{t,CAT}^3$ :

$$\arg \min_{\hat{\theta}_{t,CAT}^3} \left( F_{CAT}(t, \tau_1^3, \tau_2^3) - \int_{\tau_1^3}^{\tau_2^3} \hat{\Lambda}_u du - \hat{\mathbf{a}}_{t, \tau_1^3, \tau_2^3} \hat{\mathbf{X}}_t - \int_t^{\tau_1^1} \hat{\theta}_{t,CAT}^1 \hat{\sigma}_u \hat{\mathbf{a}}_{t, \tau_1^3, \tau_2^3} \mathbf{e}_p du - \int_{\tau_1^1}^{\tau_1^2} \hat{\theta}_{t,CAT}^2 \hat{\sigma}_u \hat{\mathbf{a}}_{t, \tau_1^3, \tau_2^3} \mathbf{e}_p du - \int_{\tau_1^3}^{\tau_2^3} \hat{\theta}_{t,CAT}^3 \hat{\sigma}_u \mathbf{e}_1^\top \mathbf{A}^{-1} [\exp \{ \mathbf{A}(\tau_2^3 - u) \} - I_p] \mathbf{e}_p du \right)^2$$

In a similar way, one obtain the estimation of  $\hat{\theta}_{t,CAT}^4, \dots, \hat{\theta}_{t,CAT}^I$ . The estimates of this parametrizations are displayed in Table 4.4.

#### 4.4.6 Smoothing the market price of risk over time

Since smoothing individual estimates is different from estimating a deterministic function, we also assure our results by fitting a parametric function to all available contract prices (calendar year estimation). After computing the MPR  $\hat{\theta}_{t,CAT}, \hat{\theta}_{t,HDD}, \hat{\theta}_{t,CDD}$  for each of the trading days for different contracts, a smoothing of the MPR with the inverse problem points can be made to find a MPR  $\hat{\theta}_u$  for every calendar day and with

#### 4 Weather Derivatives

that price temperature derivative for any date  $t \leq \tau_1^i < \tau_2^i$ ,  $i = 1, \dots, I$ :

$$\arg \min_{f \in \mathcal{F}_j} \sum_{t=1}^n \{\hat{\theta}_t - f(u_t)\}^2 = \arg \min_{\alpha_j} \sum_{t=1}^n \left\{ \hat{\theta}_t - \sum_{j=1}^J \alpha_j \Psi_j(u_t) \right\}^2 \quad (4.52)$$

where  $\Psi_j(u_t)$  is a vector of known basis functions,  $\alpha_j$  defines the coefficients,  $J$  is the number of knots,  $u_t = t - \Delta + 1$  with increment  $\Delta$  and  $n$  is the number of days to be smoothed. In our case  $u_t = 1$  day and  $\Psi_j(u_t)$  is estimated using cubic splines. Figure 3.7 shows the smoothed MPR for 1 (20060530), 5 days (20060522 - 20060530) and 30 days (20060417 - 20060530) from different parametrizations of Berlin CAT Futures. In all cases, MPR deviations are smoothed over time, fluctuating around zero and changing in sign. Alternatively, one can first do the smoothing with basis functions of all available contract prices:

$$\arg \min_{\beta_j} \sum_{t=1}^n \sum_{i=1}^I \left\{ F_{CAT}(t, \tau_1^i, \tau_2^i) - \sum_{j=1}^J \beta_j \Psi_j(u_t) \right\}^2 \quad (4.53)$$

and then obtain an estimate for  $\theta$  from the smoothed futures:

$$\begin{aligned} \arg \min_{\hat{\theta}_{t,CAT}^s} & \left( F_{CAT}^s(t, \tau_1, \tau_2) - \int_{\tau_1}^{\tau_2} \hat{\Lambda}_u du - \hat{\mathbf{a}}_{t, \tau_1, \tau_2} \mathbf{X}_t - \hat{\theta}_{t,CAT}^s \left\{ \int_t^{\tau_1} \hat{\sigma}_u \hat{\mathbf{a}}_{t, \tau_1, \tau_2} \mathbf{e}_p du \right. \right. \\ & \left. \left. + \int_{\tau_1}^{\tau_2} \hat{\sigma}_u \mathbf{e}_1^\top \mathbf{A}^{-1} [\exp \{ \mathbf{A}(\tau_2 - u) \} - I_p] \mathbf{e}_p du \right\} \right)^2 \end{aligned} \quad (4.54)$$

The last panel in Figure 4.18 shows the MPR smoothing for 1 day, 5 days and 30 days lags of the smoothed MPR estimation procedure (4.54) for CAT-Berlin Futures traded on 20060530. In both smoothing procedure cases, there are notable changes in sign, MPR deviations are smoothed over time and the higher the number of calendar days, the closer is the fit of (4.52) and (4.54).

#### 4.4.7 Statistical and economical insights of the MPR

Table 4.4 presents the median and standard deviation of different MPR parametrizations for CAT-Berlin, CAT-Essen and AAT-Tokyo futures traded during (20031006-20080527), (20050617-20090731) and (20040723-20090630) respectively with  $t \leq \tau_1^i < \tau_2^i$  (similar implications hold for other contract types).

The range of the MPR varies from -82.62 to 52.17, which corresponds to the robust estimation of the AAT-OLS2 specification, while the average MPR for (Berlin, Essen, Tokyo) daily contracts is  $(-0.00, 0.00, -3.08)$  for constant MPR,  $(-0.51, -0.38, 0.73)$  constant MPR-OLS per trading day,  $(-0.40, -0.43, -3.50)$  for OLS2-MPR,  $(0.00, 0.00, -3.08)$  for MPR spline and  $(0.00, 0.00, -0.11)$  for bootstrapping MPR. The MPR gets smaller as the measurement period  $[\tau_1, \tau_2]$  increase. In all the analyses, confirms the discussion that MPR is different from zero, it varies in time and can move from a nega-

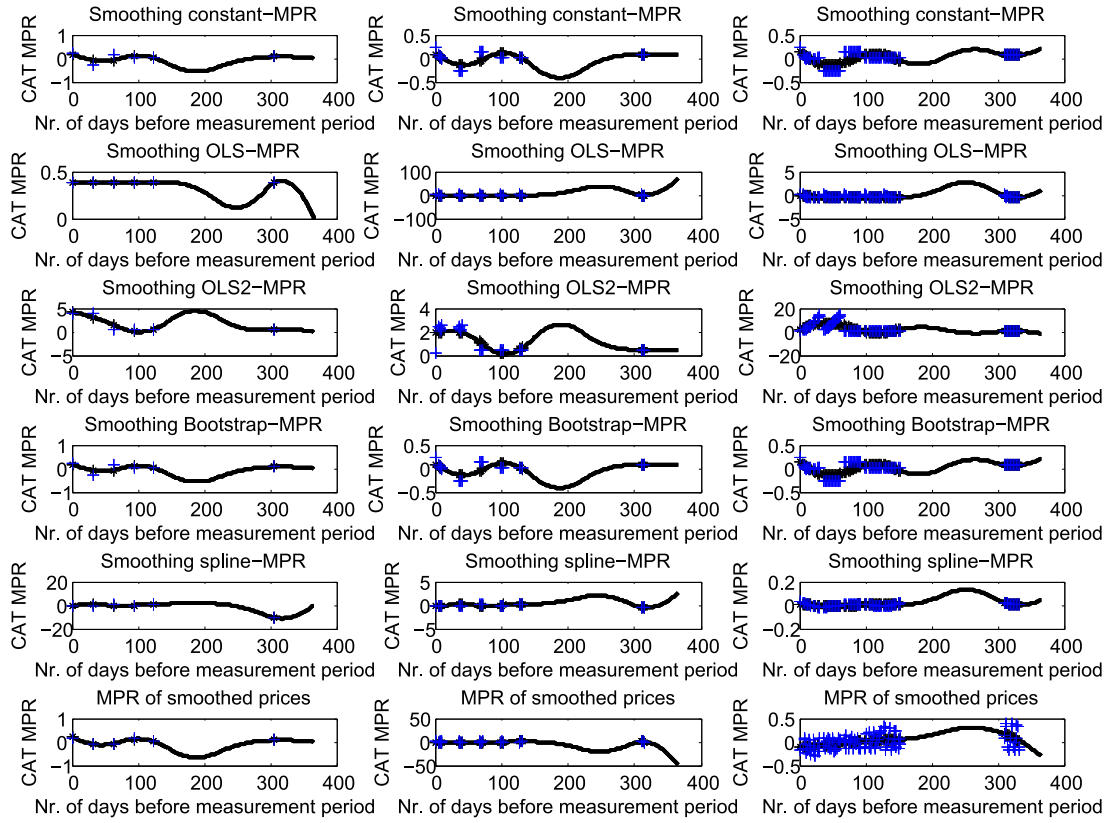


Figure 4.18: Smoothing (black line) 1 day (left), 5 days (middle), 20 days (right) of the MPR parametrization cases (gray crosses) for Berlin CAT Futures traded on 20060530. The last panel gives smoothed MPR estimates for all available contract prices.

#### 4 Weather Derivatives

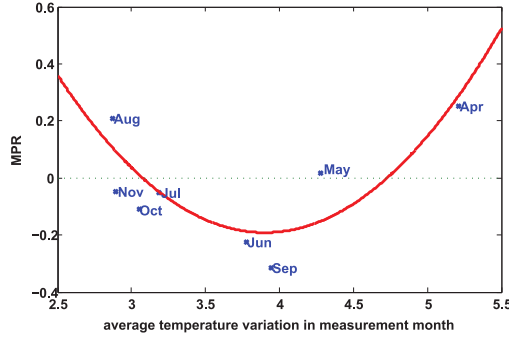


Figure 4.19: Calibrated MPR and Monthly Temperature Variation of AAT Tokyo Futures from November 2008 to November 2009 (prices for 8 contracts were available). MPR here is a nonmonotone quadratic function of  $\hat{\sigma}_{\tau_1, \tau_2}^2$ .

tive to a positive domain according to changes in the seasonal variation. For example, in Figure 4.17, the MPR sign at trading date 20060530 of a Berlin-CAT future contract changes in sign (positive - negative - positive), when switching from a 30 to 60, 90, 120, 150, 180, 210 days contract and behaves mainly negative to fast changes of the seasonal variance  $\sigma_t$  in the measurement periods.

The proposition that the MPR derived from CAT/HDD/CDD futures is different from zero is investigated. With the Wald statistical test, one can check whether this effect exists or not. In the multivariate case, the Wald statistic for  $\{\theta_t \in \mathbb{R}^p\}_{t=1}^n$  is:

$$(\hat{\theta}_t - \theta_0)^\top \Sigma (\hat{\theta}_t - \theta_0) \sim \chi_p^2, \Sigma^{\frac{1}{2}} (\hat{\theta}_t - \theta_0) \sim N(0, \mathbb{I}_p)$$

where  $\Sigma$  is the variance matrix and the estimate  $\hat{\theta}_t$  is compared with the proposed value  $\theta_0 = 0$ . Using a sample size of  $t$  trading dates of contracts with  $t \leq \tau_1^i < \tau_2^i, i = 1, \dots, I$  corresponding to  $n$  observations, the Wald statistics for all previous MPR specifications are illustrated in Table 4.4. We reject  $H_0 : \hat{\theta}_t = 0$  under the Wald statistic  $\{\theta_t \in \mathbb{R}^i\}_{t=1}^n$  for all the cases. Although, the constant and general MPR forms per trading day smooth deviations over time, under this test, we confirm that the MPR differs from zero. High MPR imply discounts for taking risk, while lower MPRs imply premiums for bearing risk.

Outside the context of the specified model and given the dependencies of the MPR on time, we believe that there should be a theoretical foundation for MPR and the temperature seasonal variation  $\sigma_t$ . In Figure 4.19, we examined the average of the calibrated  $\theta_{t,CAT}^i$  for Tokyo over the trading period  $\hat{\theta}_{\tau_1, \tau_2}^i = \sum_{t=\tau_1, \tau_2}^{\tau_2} \hat{\theta}_t^i / (\tau_2 - \tau_1, \tau_2)$  with respect to the seasonal variation in period  $[\tau_1, \tau_2]$ :  $\hat{\sigma}_{\tau_1, \tau_2}^2 = \sum_{t=\tau_1}^{\tau_2} \hat{\sigma}_t^2 / (\tau_2 - \tau_1)$ . Regressing  $\hat{\theta}_{\tau_1, \tau_2}^i$  on  $\hat{\sigma}_{\tau_1, \tau_2}^2$ , the quadratic function that parametrizes the dependence is  $\hat{\theta}_{\tau_1, \tau_2} = 4.08 + 2.19\hat{\sigma}_{\tau_1, \tau_2}^2 + 0.28\hat{\sigma}_{\tau_1, \tau_2}^4$  with  $R_{adj}^2$  equals to 0.71.

The MPR increases as the drift and volatility increases and the shape give us the insight that the effect of the MPR is nothing else than temperature variations  $\sigma_t$ . By know-

#### 4 Weather Derivatives

CAT	Berlin	Constant	OLS	OLS2	Bootstrap	Spline
WS(Prob)		0.93(0.66)	0.01(0.08)	0.00(0.04)	0.93(0.66)	0.91(0.66)
30days	Min(Max)	-0.28(0.12)	-0.65(0.11)	-1.00(0.20)	-0.28(0.12)	0.02(0.21)
(i=2)	Med(Std)	0.05(0.07)	-0.10(0.12)	-0.17(0.31)	0.05(0.07)	0.14(0.06)
60days	Min(Max)	-0.54(0.84)	-4.95(8.39)	-10.71(10.25)	-0.54(0.84)	0.00(0.21)
(i=3)	Med(Std)	0.03(0.13)	-0.09(1.06)	-0.17(1.44)	0.03(0.13)	0.05(0.05)
90days	Min(Max)	-0.54(0.82)	-4.95(8.39)	-10.71(10.25)	-0.54(0.82)	0.00(0.21)
(i=4)	Med(Std)	0.02(0.09)	-0.10(1.07)	-0.17(1.46)	0.02(0.09)	0.11(0.07)
120days	Min(Max)	-0.53(0.84)	-4.95(4.56)	-7.71(6.88)	-0.53(0.84)	0.00(0.21)
(i=5)	Med(Std)	0.03(0.14)	-0.10(0.94)	-0.19(1.38)	0.03(0.14)	0.11(0.09)
150days	Min(Max)	-0.54(0.84)	-4.53(4.56)	-8.24(6.88)	-0.54(0.84)	-0.00(0.20)
(i=6)	Med(Std)	0.13(0.09)	-0.10(0.95)	-0.20(1.47)	0.13(0.09)	0.01(0.07)
180days	Min(Max)	-0.54(0.82)	-4.53(4.56)	-8.24(6.88)	-0.54(0.82)	-0.03(0.12)
(i=7)	Med(Std)	0.02(0.12)	-0.10(0.95)	-0.14(1.48)	0.02(0.12)	0.00(0.03)
CAT	Essen	Constant	OLS	OLS2	Bootstrap	Spline
WS(Prob)		0.02(0.11)	0.20(0.34)	0.01(0.09)	0.02(0.11)	1.63(0.79)
30days	Min(Max)	-0.98(0.52)	-31.05(5.73)	-1.83(1.66)	-0.98(0.52)	-0.00(0.00)
(i=2)	Med(Std)	0.01(0.12)	-0.39(1.51)	-0.43(9.62)	0.01(0.12)	0.00(0.00)
60days	Min(Max)	-1.35(0.62)	-31.05(5.73)	-1.83(1.66)	-1.35(0.62)	0.00(0.00)
(i=3)	Med(Std)	0.02(0.14)	-0.40(1.56)	-0.46(9.99)	0.02(0.14)	0.00(0.00)
90days	Min(Max)	-1.56(0.59)	-6.68(5.14)	-5.40(1.71)	-1.56(0.59)	0.00(0.00)
(i=4)	Med(Std)	-0.02(0.19)	-0.40(0.88)	-0.40(0.85)	-0.02(0.19)	0.00(0.00)
120days	Min(Max)	-0.29(0.51)	-4.61(1.44)	-6.60(1.43)	-0.29(0.51)	0.00(0.00)
(i=5)	Med(Std)	0.03(0.05)	-0.37(0.51)	-0.41(0.85)	0.03(0.05)	0.00(0.00)
150days	Min(Max)	-0.44(0.13)	-4.61(1.44)	-6.60(1.43)	-0.44(0.13)	0.00(0.00)
(i=6)	Med(Std)	0.00(0.07)	-0.35(0.49)	-0.30(0.81)	0.00(0.07)	0.00(0.00)
180days	Min(Max)	-0.10(0.57)	-4.61(1.44)	-4.25(0.52)	-0.10(0.57)	0.00(0.00)
(i=7)	Med(Std)	-0.02(0.06)	-0.12(0.45)	-0.21(0.52)	-0.02(0.06)	0.00(0.00)
AAT	Tokyo	Constant	OLS	OLS2	Bootstrap	Spline
WS(Prob)		0.76(0.61)	0.02(0.13)	0.01(0.11)	0.76(0.61)	4.34(0.96)
30days	Min(Max)	-7.55(0.17)	-69.74(52.17)	-69.74(52.17)	-7.55(0.17)	-0.23(-0.18)
(i=2)	Med(Std)	-3.87(2.37)	-0.33(19.68)	-0.48(20.46)	-3.87(2.37)	-0.20(0.01)
60days	Min(Max)	-7.56(0.14)	-69.74(52.17)	-69.74(52.17)	-7.56(0.14)	-0.18(-0.13)
(i=3)	Med(Std)	-3.49(2.47)	-0.23( 21.34)	-0.41(21.63)	-3.49(2.47)	-0.15(0.01)
90days	Min(Max)	-7.55(1.02)	-69.74(26.82)	-69.74(38.53)	-7.55(1.02)	-0.13(-0.09)
(i=4)	Med(Std)	-2.96(2.65)	0.04(20.84)	-0.33(20.22)	-2.96(2.65)	-0.11(0.01)
120days	Min(Max)	-7.55(1.02)	-69.74(26.82)	-69.74(48.32)	-7.55(1.02)	-0.10(-0.06)
(i=5)	Med(Std)	-2.08(2.74)	1.26(19.54)	-0.11(19.69)	-2.08(2.74)	-0.08(0.01)
150days	Min(Max)	-7.55(1.02)	-51.18(26.82)	-51.18(48.32)	-7.55(1.02)	-0.06(-0.04)
(i=6)	Med(Std)	-2.08(2.71)	1.26(16.03)	7.17(17.02)	-2.08(2.71)	-0.05(0.00)
180days	Min(Max)	-7.39(1.26)	-51.18(19.10)	-51.18(48.32)	-7.39(1.26)	-0.05(-0.04)
(i=7)	Med(Std)	-2.08(2.74)	3.66(15.50)	7.63(17.69)	-2.08(2.74)	-0.04(0.00)
210days	Min(Max)	-7.39(1.26)	-24.88(26.16)	-54.14(39.65)	-7.39(1.26)	-0.07(-0.05)
(i=8)	Med(Std)	-3.00(2.86)	13.69(10.05)	10.61(14.63)	-3.00(2.86)	-0.06(0.00)
240days	Min(Max)	-7.39(1.26)	-24.88(21.23)	-82.62(42.14)	-7.39(1.26)	-0.07(-0.07)
(i=9)	Med(Std)	-3.00(2.74)	3.46(11.73)	-4.24(37.25)	-3.00(2.74)	-0.07(0.00)
270days	Min(Max)	-7.39(0.44)	-24.88(26.16)	-82.62(42.14)	-7.39(0.44)	-0.07(-0.03)
(i=10)	Med(Std)	-4.24(2.39)	8.48(13.88)	-7.39(40.76)	-4.24(2.39)	-0.05(0.00)

Table 4.4: Wald-stat (WS), Probabilities (Prob), Minimum (Min), Maximum (Max), Median (Med), Standard deviation (Std) of different MPR parametrization (Constant per contract, constant per trading date 'OLS', 2 constant per trading day 'OLS2', Bootstrap and Spline) for CAT-Berlin, CAT-Essen and AAT-Tokyo futures traded during (20031006-20080527), (20050617-20090731) and (20040723-20090630).

#### 4 Weather Derivatives

ing this formal MPR dependency and using the closest location with formal weather derivative market, the MPR for emerging regions can be inferred. In the previous example, using the Tokyo weather market, the MPR parametrization for Taipei is equal to  $\hat{\theta}_{\tau_1, \tau_2} = 4.08 - 2.19 \cdot \hat{\sigma}_{\tau_1, \tau_2}^2 + 0.28 \cdot \hat{\sigma}_{\tau_1, \tau_2}^4$ . One may also formulate that the MPR is in fact the equalizer in future temperature prices (as the implied volatilities in the option prices do). The model in (4.36) fits data extremely well and the history is likely a good prediction of the future. The dependencies of the MPR on time and temperature seasonal variation suggest that for regions with homogeneous weather risk there would be some common market price of weather risk (as we expect in equilibrium).

The non-arbitrage, risk neutral valuation paradigma induces that a risk neutral probability turns all tradable assets into martingales after discounting. In this setting all agents will act as if they are risk neutral or independent of risk preferences. We empirically argue, by using equivalent changes of measures associated with changes of drift, that the MPR differs from zero and changes in sign, revealing differences between a historical and a risk neutral behaviour of temperature. The non stationarity behaviour of the MPR (sign changes) is possible because it is capturing all the non-fundamental information affecting the future pricing: transaction costs, market illiquidity or variability, bid-ask spreads, path dependencies, investors preferences or other fractions like effects on the demand function. When the trading is illiquid the observed prices may contain some liquidity premium, which can contaminate the estimate of the MPR.

The movements of the MPR also support the existence of additional risk premium that option buyers are willing to pay for price protection and which increases the longer the measurement period. The Risk Premium (RP) is defined as a drift of the spot dynamics or a Girsanov type change of probability. In (4.28) the future price (CAT, HDD, CDD) is set under a risk neutral probability  $Q$ , thereby the risk premium measures exactly the differences between the risk neutral (observed prices) and the temperature market probability predictions:

$$RP_{(t, \tau_1, \tau_2)} = F_{(t, \tau_1^i, \tau_2^i, \theta_i)} - \hat{F}_{(t, \tau_1^i, \tau_2^i, \theta_i=0)} \quad (4.55)$$

The choice of  $Q$  determines the risk premium, and opposite, having knowledge of the risk premium determines the choice of the risk neutral probability. Negative RP means that buyers of temperature derivatives expect to pay lower prices to hedge weather risk, while positive RP indicate the existence of consumers, who consider temperature derivatives a kind of insurance. Figure 4.20 illustrates the RP of CAT-Berlin futures (the observed minus the implied future price with MPR equal to zero):

$$RP_{CAT(t, \tau_1, \tau_2)} = \int_t^{\tau_1} \theta_u \sigma_u \mathbf{a}_{t, \tau_1, \tau_2} \mathbf{e}_p du + \int_{\tau_1}^{\tau_2} \theta_u \sigma_u \mathbf{e}_1^\top \mathbf{A}^{-1} [\exp \{ \mathbf{A}(\tau_2 - u) \} - I_p] \mathbf{e}_p du$$

for contracts traded on 20031006-20080527. We observe RPs different from zero and positive (negative) MPR contributes positively (negatively) to future prices and are equivalent to positive (negative) MPR. CAT prices and CAT-RPs behave constant within the measurement month but fluctuate according the seasonal variation  $\sigma_t$  (higher prices



## 4 Weather Derivatives

City-Type	Period	$F_{\hat{\theta}_t=0}$	$F_{\hat{\theta}_t^i}$	$F_{\hat{\theta}_t^{OLS}}$	$F_{\hat{\theta}_t^{OLS2}}$	$F_{\hat{\theta}_t^{boots}}$	$F_{\hat{\theta}_t^{spl}}$
HDD-Atlanta	20061001-20070430	255.64	200.12	508.54	520.58	200.12	272.34
CAT-Berlin	20061001-20070430	21.46	34.85	129.52	129.52	34.85	21.36
CAT-Essen	20061001-20070430	22.43	52.24	456.35	457.35	52.24	22.37
AAT-Tokyo	20081001-20090430	105.08	92.61	134.37	178.37	92.61	102.82

Table 4.5: Root mean squared error (RMSE) of CAT, HDD, AAT future observed prices (Bloomberg) and future price estimates from different MPR parametrizations for contracts with  $t \leq \tau_1^i < \tau_2^i$  (MPR equal to zero  $F_{\hat{\theta}_t=0}$ , constant MPR for different contracts  $F_{\hat{\theta}_t^i}$ , constant MPR 'OLS' per trading date  $F_{\hat{\theta}_t}$ , 2 constant MPR 'OLS2'  $F_{\hat{\theta}_t^{OLS2}}$ , bootstrap MPR  $F_{\hat{\theta}_t^{boots}}$  and spline MPR  $F_{\hat{\theta}_t^{spl}}$ ).

in months with high volatilities) and a high proportion of the price value comes from the seasonal exposure. The sign of MPRs - RP change and may reflect the risk attitude and time horizon perspectives of market participants in the diversification process to hedge weather risk. The investors imputes value to the weather products, although they are nonmarketable. This might suggest some possible relationships between risk aversion and the market price of risk. Similar implications are got for HDD/CDD.

### 4.4.8 Pricing CAT-HDD-CDD futures

Once that the MPR is known for temperature futures, the MPR for options is also known and thus one can price new non-standard maturities derivatives. The prices of the HDD futures  $\hat{F}_{t,\tau_1,\tau_2,HDD}$  for Berlin, Essen, Tokyo and Osaka can be calculated with the extracted CAT-MPR estimated under different parametrizations, from the HDD-CDD parity Equation (4.4), one might get the price for HDD futures and compare this with the actual observed prices value. When the implied future prices do not equal the observed prices, we say that the temperature market probability predictions operate under more general equilibrium rather than no-arbitrage conditions. The root squared meand errors (RMSE) between observed prices (Bloomberg) and the estimates are shown in Table 4.5, the estimates with constant MPR are statistically equivalent enough to the known CAT future prices. HDD future are not fully replicated with the information extracted given by the MPR  $\theta_t$  of CAT futures. Hence, there is some added value that the market incorporates to the HDD prices estimation. It might be due to sophisticated information of weather forecast models introduce to the pricing model (see e.g. Benth and Meyer-Brandis [2009]) or changes in the risk attitude of market participants in the diversification process of hedging weather risk.

## 4.5 The risk premium and the market price of weather risk

Since appropriate measures of the risk associated to a particular price become necessary for pricing, one essentially needs to incorporate the market price of risk (MPR), which is an important parameter of the associated equivalent martingale measure. The market

#### 4 Weather Derivatives

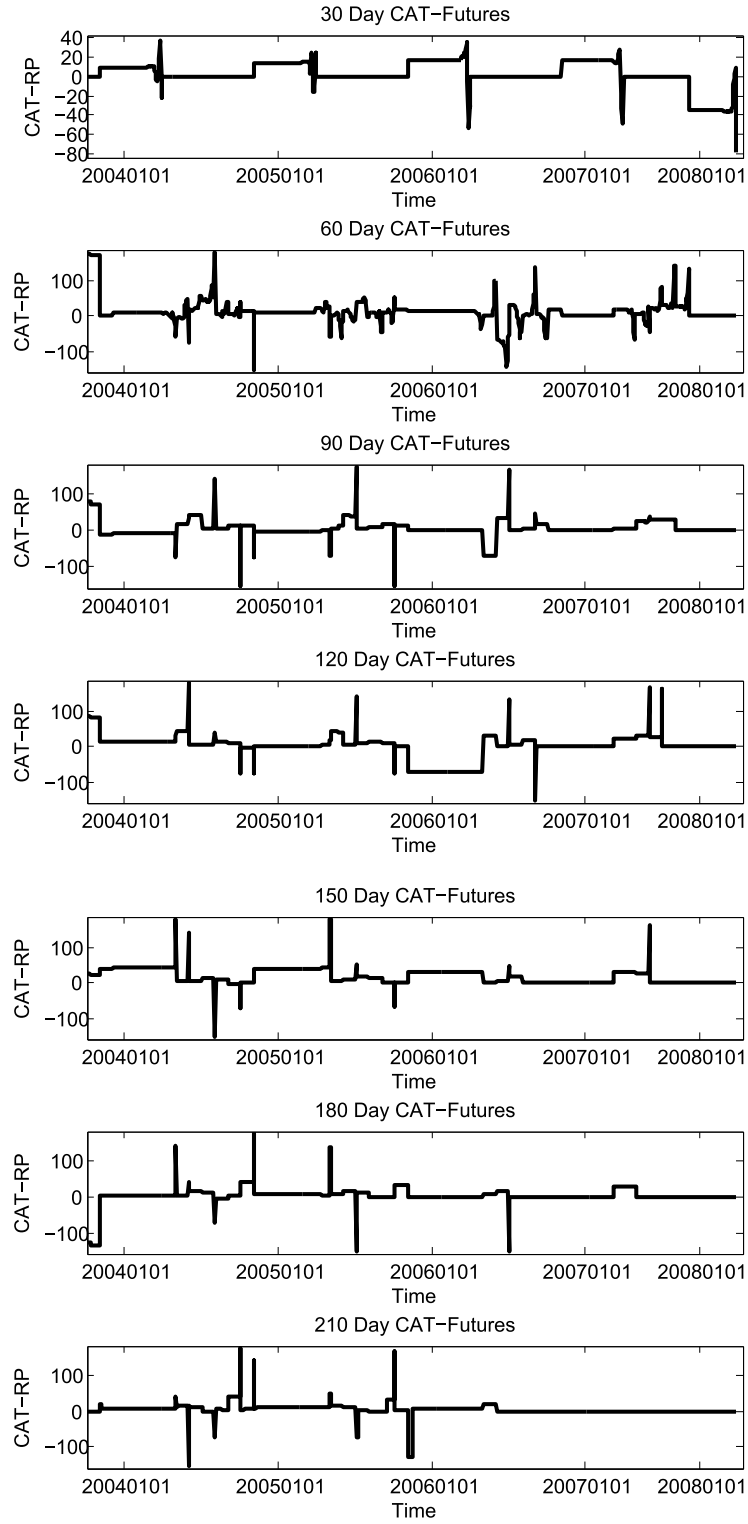


Figure 4.20: Risk premiums (RP) of CAT-Berlin future prices traded during (20031006 - 20080527) with  $t \leq \tau_1^i < \tau_2^i$ .

#### 4 Weather Derivatives

price of temperature risk can be seen as the drift adjustment in the dynamics of temperature and it reflects the compensation for bearing risk with a temperature derivative.

As it has been seen in 4.4 the MPR may be positive or negative depending on the time horizon considered. Research on other energy markets has reflected that the MPR for electricity markets is negative (Cartea and Figueroa. [2005]), while for gas markets a positive MPR is observed for long term contracts and changing signs across time for short term contracts (Cartea and Williams [2007]). We further extend the empirical behavior of the true MPR from last section to a model specifications that cover a possible stochasticity in the observed MPR/RP. By Girsanov Theorem, the MPR can be parametrized via a class of risk neutral probabilities.

Suppose that the future price is defined as in (4.28). The probability measure  $Q_\theta$  is a Girsanov transform of the Brownian motion  $B_t$ :

$$dB_t^\theta = dB_t - \theta_t \sigma_t dt \quad (4.56)$$

where  $B_t^\theta$  is a Brownian motion on  $[0, T]$ . Let us consider a measure change (a  $Q_\theta$  probability) such that the AR-coefficients in the CAR( $p$ ) model are changed:

$$dX_{1(t)} = \theta_t dt + \sigma_t dB_t^\theta \quad (4.57)$$

The explicit representation of  $X_t$  under  $Q_\theta$  is:

$$\mathbf{X}_s = \mathbf{x} + \int_t^s \theta_u du + \int_t^s \sigma_u dB_u^\theta$$

where  $\mathbf{X}_t = x$ . The future prices are now given by:

$$\begin{aligned} F_{HDD}(t, \tau_1, \tau_2) &= \int_{\tau_1}^{\tau_2} v_{t,s} \psi \left[ \frac{c - m_{\{t,s,\mathbf{X}_t\}}}{v_{t,s}} \right] ds \\ F_{CDD}(t, \tau_1, \tau_2) &= \int_{\tau_1}^{\tau_2} v_{t,s} \psi \left[ \frac{m_{\{t,s,\mathbf{X}_t\}} - c}{v_{t,s}} \right] ds \\ F_{CAT}(t, \tau_1, \tau_2) &= \int_{\tau_1}^{\tau_2} \Lambda_u du + \mathbf{X}_t + \int_{\tau_1}^{\tau_2} \theta_u du + \int_{\tau_1}^{\tau_2} \sigma_u du \end{aligned} \quad (4.58)$$

$$\begin{aligned} m_{\{t,s,x\}} &= \Lambda_s + \int_t^s \theta_u du + x \\ v_{t,s}^2 &= \int_t^s \sigma_u^2 du \end{aligned} \quad (4.59)$$

with  $\psi(x) = x\Phi(x) + \varphi(x)$ . This gives much flexibility when creating different risk premium structures. Then the market risk premium is positive as long as:

$$RP_{(t,\tau_1,\tau_2)} = \int_{\tau_1}^{\tau_2} \theta_u du \quad (4.60)$$

#### 4 Weather Derivatives

is positive. By specifying  $\theta_t$ , one can obtain a change in the sign of the market risk premium over time.

In temperature markets, the spot-future relation is not clear since the underlying is not storable. One may explain this relation by defining a specification of the risk neutral probability (the market price of risk). Alternatively to explaining future prices by the underlying spot, one may use the Heath-Jarrow-Morton (HJM) approach from interest rate theory, Heath [1992] which assumes dynamics for the future price evolution. This can be done in terms of market dynamics or risk neutral measure. Once having these risk neutral dynamics, one can calculate the conditional expectation of the pay-off from the option and hedging can be done.

Let  $r$  be the constant risk free interest rate and let  $\hat{w}_u = 1$  be a weight function if the temperature future (4.28) is settled at the end of the delivery period  $\tau_2$  or  $\hat{w}_u = \exp(-ru)$  if the contract is settled continuously over the delivery period. Then, a weight that integrates to one is:

$$\int_s^t w_{(u,s,t)} du = \int_s^t \frac{\hat{w}_u}{\int_s^t \hat{w}_v dv} du = 1 \quad (4.61)$$

where  $0 \leq u \leq s < t$ . Note that  $w_{(u,s,t)} = \frac{1}{t-s}$  when  $\hat{w}_u = 1$ , and  $w_{(u,s,t)} = \frac{r \exp(-ru)}{\exp(-rs) - \exp(-rt)}$  when  $\hat{w}_u = \exp(-ru)$ . Then a link for the future and the underlying temperature process is:

$$F_{(t,\tau_1,\tau_2)} = E^Q \left[ \int_{\tau_1}^{\tau_2} w_{(u,\tau_1,\tau_2)} T_u du \middle| \mathcal{F}_t \right] \quad (4.62)$$

where  $Q$  is an equivalent martingale measure. Commuting the conditional expectation with Lebesgue integration in (4.62), we get that  $E^Q \left[ \int_{\tau_1}^{\tau_2} |w_{(u,\tau_1,\tau_2)} T_u| du \right] < \infty$ . It holds then,

$$F_{(t,\tau_1,\tau_2)} = \int_{\tau_1}^{\tau_2} w_{(u,\tau_1,\tau_2)} f_{(t,u)} du \quad (4.63)$$

where  $f_{(t,u)}$  denotes the price of a forward prices with a future delivery time  $\tau$  with  $0 \leq t \leq \tau < \infty$ . This relationship means that a future contract with deliver over a period can be considered as a weighted stream of forwards. and at delivery the future price coincides with the price of temperature  $T_t$ . Supposing that  $E^Q [|S_\tau|] < \infty$  and time approaches delivery, then there is convergence of future prices to the spot prices:

$$\lim_{t \rightarrow \tau} f(t, \tau) = f(\tau, \tau) = E^Q [T_\tau | \mathcal{F}_\tau] = T_\tau \quad (4.64)$$

because  $f(t, \tau)$  does not have fixed time discontinuities and the dynamics of the  $T_\tau$  is  $\mathcal{F}_\tau$ -measurable. Nevertheless, the convergence of future prices to the spot at delivery

## 4 Weather Derivatives

does not hold:

$$\begin{aligned}
 \lim_{t \rightarrow \tau_1} F(t, \tau_1, \tau_2) &= \lim_{t \rightarrow \tau_1} \int_{\tau_1}^{\tau_2} w_{(u, \tau_1, \tau_2)} f_{(t, u)} du \\
 &= \int_{\tau_1}^{\tau_2} w_{(u, \tau_1, \tau_2)} \lim_{t \rightarrow \tau_1} f_{(t, u)} du \\
 &= \lim_{t \rightarrow \tau_1} \int_{\tau_1}^{\tau_2} w_{(u, \tau_1, \tau_2)} f_{(\tau_1, u)} du
 \end{aligned} \tag{4.65}$$

which is different from  $S_{\tau_1}$  a.s. as long the  $S_t$  is not a  $Q$ -Martingale. Note that if  $\hat{w}_u = 1$  and  $F_{(\tau_1, \tau_1, \tau_2)} = S_{\tau_1}$  it holds:

$$\int_{\tau_1}^{\tau_2} f_{(\tau_1, u)} du = (\tau_2 - \tau_1) S_{\tau_1} \tag{4.66}$$

Taking the derivative with respect to  $\tau_2$ , we get:

$$\begin{aligned}
 f_{(\tau_1, \tau_2)} &= S_{\tau_1} \\
 E_Q[S_{\tau_2} | \mathcal{F}_{\tau_1}] &= S_{\tau_1}
 \end{aligned} \tag{4.67}$$

$S_t$  must be martingale under  $Q$ . If we assume that the delivery period collapses into a single point, then the future contract is similar to the forward:

$$\begin{aligned}
 \lim_{\tau_2 \rightarrow \tau_1} F_{(\tau_1, \tau_1, \tau_2)} &= \lim_{\tau_2 \rightarrow \tau_1} \int_{\tau_1}^{\tau_2} w_{(u, \tau_1, \tau_2)} f_{(t, u)} du \\
 &= \lim_{\tau_2 \rightarrow \tau_1} \frac{\int_{\tau_1}^{\tau_2} \hat{w}_{(u)} f_{(\tau_1, u)} du}{\int_{\tau_1}^{\tau_2} \hat{w}_{(u)} du} \\
 &= f_{(t, \tau_1)}
 \end{aligned} \tag{4.68}$$

Consider the future price  $F_{(t, \tau_1, \tau_N)}$  with delivery over  $[\tau_1, \tau_N]$  (the seasonal strips contracts) and  $N$  contracts  $t, \tau_k, \tau_{k+1}$  with delivery over  $[\tau_k, \tau_{k+1}]$ ,  $k = 1, \dots, N-1$ . Assuming that  $\tau_1 < \tau_2 < \dots < \tau_N$ , then the no arbitrage relationship between traded future prices is:

$$F_{(t, \tau_1, \tau_N)} = \sum_{k=1}^{N-1} w_k F_{(t, \tau_k, \tau_{k+1})} \tag{4.69}$$

with

$$w_k = \frac{\int_{\tau_k}^{\tau_{k+1}} w_u du}{\int_{\tau_1}^{\tau_N} w_u du} \tag{4.70}$$

This is the discrete version of (4.63) if  $N \rightarrow \infty$  and  $\tau_k = \tau_1 + (k-1) \times \Delta$  with  $\Delta = \frac{\tau_2 - \tau_1}{N}$ . Since the dynamics of future prices become intractable as in section 4.4, this approach

can be applied to derive forward contracts from futures. By smoothing the historical future curve, we can estimate the market price of risk. Under this setting, (4.55) become:

$$RP_\tau = f_{(t,\tau)} - E[S_\tau | \mathcal{F}_\tau] \quad (4.71)$$

and we might be able to obtain changes in sign of the market price premium.

## 4.6 Temperature baskets

Cao and Wei [2004] and Benth and Benth [2005] observed that temperature variations are correlated between locations. The closer the locations are geographically, the greater the correlation between the daily average temperatures. The correlation of temperature between pair of stations in Lithuania has been studied in Benth et al. [2007a].

In the next section we construct basket temperatures of daily average temperature. Rather than purchasing temperature derivatives at each city, it would be more cost efficient to purchase a basket temperature index for the cities in question. The goal of this section is to hedge weather risk in more than one location. The weightings of the basket will depend on the exposure to temperature risk that every location face. Our model extends the model in Benth and Benth [2005] to  $n$ -dimension.

### 4.6.1 Basket indices

A temperature basket is a weighted sum of the daily average temperatures of  $n$  spatial locations. Let  $A_\tau$  be the temperature basket, defined as:

$$A_\tau = \sum_{i=1}^n w_i T_{i(\tau)} \quad (4.72)$$

where  $w_i$  is the weight allocated to location  $x_i$  and  $\sum_{i=1}^n w_i = 1$ . Suppose that the baskets are set under a continuous time setting. We construct temperature baskets for HDD, CDD, CAT, C24AT indices. Let  $HDD_A, CDD_A, CAT_A$  be the basket index over  $[\tau_1, \tau_2]$  given by:

$$\begin{aligned} HDD_A(\tau_1, \tau_2) &= \int_{\tau_1}^{\tau_2} \max(c - A_\tau, 0) d\tau \\ CDD_A(\tau_1, \tau_2) &= \int_{\tau_1}^{\tau_2} \max(A_\tau - c, 0) d\tau \\ CAT_A(\tau_1, \tau_2) &= \int_{\tau_1}^{\tau_2} A_\tau d\tau \\ C24AT_A(\tau_1, \tau_2) &= \int_{\tau_1}^{\tau_2} A_\tau d\tau \end{aligned} \quad (4.73)$$

## 4 Weather Derivatives

where  $c$  is a baseline temperature. Similar as (4.4), we have:

$$CDD_A(\tau_1, \tau_2) - HDD_A(\tau_1, \tau_2) = CAT_A(\tau_1, \tau_2) - c(\tau_2 - \tau_1) \quad (4.74)$$

### 4.6.2 Stochastic modelling for Basket temperatures

Let the probability space  $(\Omega, \mathcal{F}, P)$ . Let  $T_i$  be the daily average temperature at location  $x_i$  modelled as in (4.30):

$$d\mathbf{X}_t = \mathbf{A}\mathbf{X}_t dt + \mathbf{e}_p \sigma_t dB_t \quad (4.75)$$

with  $\sigma_t$  defined in (4.21). Consider two standard Brownian motions  $B_{i(t)}$  and  $B_{j(t)}$  denoting the random noise component of daily average temperatures at location  $x_i$  and  $x_j$ , correlated spatially with covariance:

$$\text{Cov}(B_{i(t)}, B_{j(t)}) = \min(t, s)q(|x_i - x_j|) \quad (4.76)$$

where  $q$  is a monotonic decreasing towards zero function of the distance between two locations. The corresponding correlation is given by:

$$dB_{i(t)}dB_{j(t)} = q(|x_i - x_j|)dt = \rho_{i,j}dt \quad (4.77)$$

Meaning that if  $q(|x_i - x_i|) = q(0) = \rho_{i,i} = 1$ ,  $q(|x_i - x_j|) = q(|x_j - x_i|) = \rho_{i,j} = \rho_{j,i}$ . Considering (4.30) for  $i = 1, \dots, n$  locations, we have:

$$d\mathbf{Z}_t = \mathbf{E}\mathbf{Z}_t dt + \Sigma_t d\mathbf{B}_t \quad (4.78)$$

where

$$\mathbf{Z}_t = \begin{pmatrix} \mathbf{X}_{1(t)} \\ \mathbf{X}_{2(t)} \\ \vdots \\ \mathbf{X}_{n(t)} \end{pmatrix}, \mathbf{B}_t = \begin{pmatrix} B_{1(t)} \\ B_{2(t)} \\ \vdots \\ B_{n(t)} \end{pmatrix}, \mathbf{E} = \begin{pmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \\ \vdots \\ \mathbf{A}_n \end{pmatrix}, \quad (4.79)$$

$$\Sigma_t = \begin{pmatrix} \mathbf{e}_{1(p)}\sigma_{1(t)} & 0 & \dots & 0 \\ 0 & \mathbf{e}_{2(p)}\sigma_{2(t)} & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & \mathbf{e}_{n(p)}\sigma_{n(t)} \end{pmatrix} \quad (4.80)$$

## 4 Weather Derivatives

where  $\mathbf{B}_t \sim \mathbf{N}(0, \Omega t)$  and  $\Omega$  is a covariance matrix:

$$\Omega_t = \begin{pmatrix} 1 & \rho_{12} & \dots & \rho_{1n} \\ \rho_{21} & 1 & \dots & \rho_{2n} \\ \vdots & \vdots & \dots & \vdots \\ \rho_{n1} & \rho_{n2} & \dots & 1 \end{pmatrix} \quad (4.81)$$

### 4.6.3 Pricing of Basket temperatures

The daily average temperatures at each location is modelled as in (4.30), with random noise term correlated spatially. The risk neutral temperature distribution is normal, and hence the basket is a weighted sum of normal distributions which is also normally distributed. For the case when the distribution is unknown, one can use a moment matching principle to price basket derivatives, see Brigo et al. [2004].

The expected discounted value approach is recalled to price a temperature basket. Under the risk neutral probability  $Q_\theta$  at time  $t \leq \tau_1$  for the measurement period  $[\tau_1, \tau_2]$ , the value of a future on basket index is:

$$F_{(t, \tau_1, \tau_2, A)} = E^{Q_\theta} [Y_A | \mathcal{F}_t] \quad (4.82)$$

where  $Y_A$  refers to the random payoff at expiry for the owner of a future contract on the basket index. Let  $\Theta = (\theta_{1(t)}, \theta_{2(t)}, \dots, \theta_{n(t)})$  be an adapted process in  $[0, \tau_{\max}]$

By Cholesky factorization  $\mathbf{W}_t = \mathbf{L}\mathbf{B}_t$ , where  $\mathbf{L}$  is a Brownian motion and  $\mathbf{L}\mathbf{L}^\top = \Omega$ , and

$$\mathbf{B}_t = \begin{pmatrix} B_{1(t)} \\ B_{2(t)} \\ \vdots \\ B_{n(t)} \end{pmatrix}, dB_{i(t)}, dB_{j(t)} = \delta_{ij} dt$$

Then (4.78) becomes:

$$d\mathbf{Z}_t = \mathbf{E}\mathbf{Z}_t dt + \Sigma_t \mathbf{L} d\mathbf{B}_t \quad (4.83)$$

By recalling that  $B_{i(t)}$  are independent Brownian motions, the Girsanov transformation says that:

$$\mathbf{B}_t^\Theta = \mathbf{B}_t - \int_0^t (\Sigma_u \mathbf{L})^{-1} \Theta_u du \quad (4.84)$$

is a standard Brownian motion under the equivalent martingale measure  $Q_\theta$ . Then, the dynamics in (4.78) will become:

$$d\mathbf{Z}_t = (\mathbf{E}\mathbf{Z}_t + \Theta_t) dt + \Sigma_t \mathbf{L} d\mathbf{B}_t^\Theta \quad (4.85)$$



#### 4 Weather Derivatives

with explicit dynamics, for  $t \geq 0$ :

$$\mathbf{Z}_t = \exp \{ \mathbf{E}(t) \} \mathbf{z} + \int_0^t \exp \{ \mathbf{E}(t-u) \} \Theta_u du + \int_0^t \exp \{ \mathbf{E}(t-u) \} \Sigma_u \mathbf{L} d\mathbf{B}_u^\Theta$$

where  $\mathbf{Z}_t = \mathbf{z} \in \mathbb{R}^{p \times n}$ . The  $i$ th row denotes the temperature at location  $x_i$  as:

$$\begin{aligned} \mathbf{X}_{i(t)} &= \exp \{ \mathbf{A}_i(t) \} \mathbf{x} + \int_0^t \exp \{ \mathbf{A}_i(t-u) \} \mathbf{e}_{i(p)} \theta_{i(u)} du \\ &\quad + \int_0^t \exp \{ \mathbf{A}_i(t-u) \} \mathbf{e}_{i(p)} \sigma_{i(u)} \sum_{j=1}^i \mathbf{L}_{ij} dB_{j(u)}^\theta \end{aligned} \quad (4.86)$$

Inserting (4.72) and (4.73) in (4.82), we have that the future price is:

$$\begin{aligned} F_{CAT}(t, \tau_1, \tau_2, A) &= \mathbb{E}^{Q_\theta} \left[ \sum_{i=1}^n w_i \left\{ \int_{\tau_1}^{\tau_2} T_{i(\tau)} d\tau \right\} \middle| \mathcal{F}_t \right] \\ &= \sum_{i=1}^n w_i \mathbb{E}^{Q_\theta} \left[ \left\{ \int_{\tau_1}^{\tau_2} T_{i(\tau)} d\tau \right\} \middle| \mathcal{F}_t \right] \end{aligned} \quad (4.87)$$

Observe that:

$$\begin{aligned} T_{i(\tau)} &= \Lambda_{i(\tau)} + \exp \{ \mathbf{A}_i(\tau-t) \} \mathbf{x} + \int_t^\tau \exp \{ \mathbf{A}_i(\tau-u) \} \mathbf{e}_{i(p)} \theta_{i(u)} du \\ &\quad + \int_t^\tau \exp \{ \mathbf{A}_i(\tau-u) \} \mathbf{e}_{i(p)} \sigma_{i(u)} dB_{i(u)}^\theta \end{aligned} \quad (4.88)$$

Integrating from  $\tau_1$  to  $\tau_2$  and taking expectations, we have that:

$$\begin{aligned} \mathbb{E}^{Q_\theta} \left[ \int_{\tau_1}^{\tau_2} T_{i(\tau)} d\tau \middle| \mathcal{F}_t \right] &= \int_{\tau_1}^{\tau_2} \Lambda_{i(\tau)} d\tau + \mathbf{a}_{i(t, \tau_1, \tau_2)} \mathbf{X}_{i(t)} \\ &\quad + \int_t^{\tau_1} \theta_{i(u)} \mathbf{a}_{i(t, \tau_1, \tau_2)} \mathbf{e}_{i(p)} du \\ &\quad + \int_{\tau_1}^{\tau_2} \theta_{i(u)} \mathbf{e}_{i(1)}^\top \mathbf{A}_i^{-1} \left[ \exp \{ \mathbf{A}_i(\tau_2-u) \} - I_{i(p)} \right] \mathbf{e}_{i(p)} du \end{aligned} \quad (4.89)$$

with

$$\mathbf{a}_{i(t, \tau_1, \tau_2)} = \mathbf{e}_{i(1)}^\top \mathbf{A}_i^{-1} [\exp \{ \mathbf{A}_i(\tau_2-t) \} - \exp \{ \mathbf{A}_i(\tau_1-t) \}]$$

and  $I_{i(p)}$ - $p \times p$  identity matrix. By Itô's Lemma the expectation of the last term in (4.88) does not contribute to the expectation in (4.89).

#### 4 Weather Derivatives

Combining (4.87) and (4.89):

$$\begin{aligned}
F_{CAT}(t, \tau_1, \tau_2, A) = & \sum_{i=1}^n w_i \left( \int_{\tau_1}^{\tau_2} \Lambda_{i(\tau)} d\tau + \mathbf{a}_{i(t, \tau_1, \tau_2)} \mathbf{X}_{i(t)} \right. \\
& + \int_t^{\tau_1} \theta_{i(u)} \mathbf{a}_{i(t, \tau_1, \tau_2)} \mathbf{e}_{i(p)} du \\
& \left. + \int_{\tau_1}^{\tau_2} \theta_{i(u)} \mathbf{e}_{i(1)}^\top \mathbf{A}_i^{-1} \left[ \exp \{ \mathbf{A}_i(\tau_2 - u) \} - I_{i(p)} \right] \mathbf{e}_{i(p)} du \right) \quad (4.90)
\end{aligned}$$

Note that the only coordinate on  $\mathbf{X}_{i(t)}$  that has a  $dB_{i(u)}^\theta$  term is  $X_{i(t)}$  and since the future price  $F_{CAT}(t, \tau_1, \tau_2, A)$  is a  $Q^\theta$ -martingale, after applying the multidimensional Itô's Formula, the dynamics are:

$$dF_{CAT}(t, \tau_1, \tau_2, A) = \sum_{i=1}^n w_i \Sigma_{i, CAT}(t, \tau_1, \tau_2, A) dB_{i(t)}^\theta \quad (4.91)$$

where

$$\Sigma_{i, CAT}(t, \tau_1, \tau_2, A) = \sigma_{i(t)} \mathbf{e}_{i(1)}^\top \mathbf{A}_i^{-1} [\exp \{ \mathbf{A}_i(\tau_2 - t) \} - \exp \{ \mathbf{A}_i(\tau_1 - t) \}] \mathbf{e}_{i(p)}$$

The price of a call option written on  $F_{CAT}(t, \tau_1, \tau_2, A)$  is given by:

$$C_{CAT}(t, \tau, \tau_1, \tau_2, K, A) = \exp \{ -r(\tau - t) \} E^{Q^\theta} \left[ \max(F_{CAT}(t, \tau_1, \tau_2, A) - K, 0) | \mathcal{F}_t \right] \quad (4.92)$$

But

$$\int_t^\tau dF_{CAT}(u, \tau_1, \tau_2, A) = F_{CAT}(\tau, \tau_1, \tau_2, A) - F_{CAT}(t, \tau_1, \tau_2, A) \quad (4.93)$$

and by (4.91),

$$F_{CAT}(\tau, \tau_1, \tau_2, A) = F_{CAT}(t, \tau_1, \tau_2, A) + \sum_{i=1}^n w_i \int_t^\tau \Sigma_{i, CAT}(u, \tau_1, \tau_2, A) dB_{i(u)}^\theta$$

In order to compute (4.92), we need:

$$\sum_{i=1}^n w_i \int_t^\tau \Sigma_{i, CAT}(u, \tau_1, \tau_2, A) dB_{i(u)}^\theta > K - F_{CAT}(t, \tau_1, \tau_2, A)$$

The distribution of the dynamics is normal distributed, with mean zero and variance:

$$w_i^2 \int_t^\tau \Sigma_{i, CAT}^2(s, \tau_1, \tau_2, A) ds$$

as the weighted sum of normal random variables is again normal distributed. More-

## 4 Weather Derivatives

over, Let  $Z_i = w_i \int_t^\tau \Sigma_{i,CAT(u,\tau_1,\tau_2,A)} dB_{i(u)}^\theta$ , then:

$$\begin{aligned} \text{Cov}(Z_i, Z_j) &= w_i w_j \int_t^\tau \Sigma_{i,CAT(u,\tau_1,\tau_2,A)} du, j > i \\ \Pi_{t,\tau}^2 &= \text{Var}\left(\sum_i^n Z_i\right) \\ &= \sum_{i=1}^n \text{Var}(Z_i) + 2 \sum_{i,j} \text{Cov}(Z_i, Z_j) + \sum_{i=1}^n w_i^2 \int_t^\tau \Sigma_{i,CAT(u,\tau_1,\tau_2,A)}^2 du \\ &\quad + 2 \sum_{i=1}^n w_i w_j \int_t^\tau \Sigma_{j,CAT(u,\tau_1,\tau_2,A)} du \end{aligned} \quad (4.94)$$

Hence then  $\sum_{i=1}^n Z_i \sim \mathbf{N}(0, \Pi_{t,\tau}^2)$ . Therefore, in (4.94) it follows that:

$$\Pi_{t,\tau}^2 X > K - F_{CAT(t,\tau_1,\tau_2,A)}, X \sim \mathbf{N}(0, 1) \quad (4.95)$$

Then

$$X > \frac{K - F_{CAT(t,\tau_1,\tau_2,A)}}{\Pi_{t,\tau}^2} = d_{(t,\tau)} \quad (4.96)$$

Then the explicit formulae for the CAT call option (4.92) written on a CAT future with strike  $K$  at exercise time  $\tau < \tau_1$  during the period  $[\tau_1, \tau_2]$  is:

$$\begin{aligned} C_{CAT(t,\tau,\tau_1,\tau_2,K,A)} &= \exp\{-r(\tau - t)\} \left( F_{CAT(t,\tau_1,\tau_2,A)} - K \right) \Phi\left\{d_{(t,\tau)}\right\} \\ &\quad + \frac{\Pi_{t,\tau}^2}{\sqrt{2\pi}} \exp\{-r(\tau - t)\} \exp\left\{-\frac{1}{2}d_{(t,\tau)}^2\right\} \end{aligned} \quad (4.97)$$

where  $d_{(t,\tau)} = \frac{F_{CAT(t,\tau_1,\tau_2,A)} - K}{\Pi_{t,\tau}}$ . Note that there is no MPR  $\Theta$  term in (4.97) because the option can be perfectly hedged using  $F_{CAT(t,\tau_1,\tau_2,A)}$ . In order to pin down the basket CAT future price, the market price of risk for each of the locations need to be specified. The MPR can be estimated by calibrating to today's observed CAT future curves. The future prices for the C24AT, HDD and CDD index can be similarly estimated.

## 4.7 Conclusions and further research

This chapter deals with the differences between historical and risk neutral behaviors of temperature and gives insights into the market price of weather risk - MPR (change of drift). The empirical results shows that independently of the chosen location, the temperature driving stochastics are close to a Wiener Process that allows us to work under the financial mathematical context.

Using statistical modelling, the MPR is implied from daily temperature futures type contracts (CAT, CDD, HDD and AAT) traded at the Chicago Mercantile Exchange

(CME). This chapter empirically investigates how the MPR behaves in reality and as should be in theory. Different specifications of the MPR are investigated. It can be parameterized, given its dependencies on time and temperature seasonal variation. We also establish connections between the market risk premium (RP) and the MPR. The results show that the MPRs - RP are significantly different from zero, changing over time. In particular, the sign changes are determined by the risk attitude and time horizon perspectives of market participants in the diversification process to hedge weather risk and their effect on the demand function. This brings significant challenges to the statistical branch of the pricing literature, suggesting that for regions with homogeneous weather risk there is a common market price of weather risk.

A further research on the explicit relationship between the RP and the MPR should be carried to explain possible connections between modelled future prices and their deviations from the future market. Some efforts should be done on the model specification for the MPR to avoid arbitrage opportunities. An important issue for our results is that the econometric part in Section 2 is carried with estimates rather than true values. One thus deals with noisy observations, which are likely to alter the subsequent estimation and test procedure. An alternative to this is to use an adaptive local parametric estimation procedure, for example in Cizek et al. [2009] or Mercurio and Spokoiny [2004].

The pricing of temperature baskets was also studied. However, an extension would be to price rainbow options (maximum/minimum type basket options) at a number of locations. A discussion of a spatial temporal pricing model becomes necessary, as investors are exposed to temperature risk over a region rather than a location, where there is no formal weather market. It is therefore important to understand a temperature dependency between locations, see Benth et al. [2007a]. Finally, a different methodology, but related to the topic, would be to imply the pricing kernel of option prices.

# Bibliography

- P. Alaton, B. Djehiche, and D. Stillberger. On modelling and pricing weather derivatives. *Appl. Math. Finance*, 9(1):1–20, 2002.
- R. Anderson, F. Bendimerad, E. Canabarro, and M. Finkemeier. Analyzing insurance-linked securities. *Journal of Finance*, 1(2):49–78, 2000.
- P. Barrieu and N. El Karoui. Optimal design of weather derivatives. *ALGO Research*, 5(1), 2002.
- P. Barrieu and H. Loubergé. Hybrid cat-bonds. *Journal of Risk and Insurance*, 76(3): 547–578, 2009.
- Y. Baryshnikov, A. Mayo, and D. R. Taylor. Pricing of cat bonds. *Preprint*, 2001.
- F.E. Benth. On arbitrage-free pricing of weather derivatives based on fractional brownian motion. *Appl. Math. Finance*, 10(4):303–324, 2003.
- F.E. Benth and S. Benth. Stochastic modelling of temperature variations with a view towards weather derivatives. *Appl. Math. Finance*, 12(1):53–85, 2005.
- F.E. Benth and S. Benth. The volatility of temperature and pricing of weather derivatives. *Quantitative Finance*, 7(5):553–561, 2007.
- F.E. Benth and T. Meyer-Brandis. The information premium for non-storable commodities. *Journal of Energy Markets*, 2(3), 2009.
- F.E. Benth, S. Benth, and P. Jalinska. A spatial-temporal model for temperature with seasonal variance. *Applied Statistics*, 34(7):823–841, 2007a.
- F.E. Benth, S. Benth, and S. Koekebakker. Putting a price on temperature. *Scandinavian Journal of Statistics*, 34:746–767, 2007b.
- F.E. Benth, S. Benth, and S. Koekebakker. *Stochastic Modelling of Electricity and Related Markets*. World Scientific. Advanced Series on Statistical Science and Applied Probability, ii edition, 2008.
- F.E. Benth, W. K. Haerdle, and B. López Cabrera. *Pricing Asian temperature Risk in Statistical Tools for Finance and Insurance* (P. Cizek, W. Haerdle and R. Weron, eds.). Springer Verlag Heidelberg, 2010.
- T. Bjork. *Arbitrage Theory in Continuous Time*. Oxford University Press, Oxford, 1998.

## Bibliography

- D. Brigo, F. Mercurio, F. Rapisarda, and R. Scotti. Approximated moment matching dynamics for basket option pricing. *Quantitative Finance*, 4(1):1–16, 2004.
- D.C. Brody, J. Syroka, and M. Zervos. Dynamical pricing of weather derivatives. *Quantitative Finance*, 2(3):189–198, 2002.
- K. Burnecki and G. Kukla. Pricing of zero-coupon and coupon cat bonds. *Appl. Math.*, 30:315–324, 2003.
- K. Burnecki, G. Kukla, and R. Weron. Property insurance loss distributions. *Physica A*, 287:269–278, 2000.
- K. Burnecki, W. K. Haerdle, and R. Weron. *Simulation of risk processes in Encyclopedia of Actuarial Science Volume X*, (J. Teugels and B. Sundt, eds.). Wiley, Chichester, 2004.
- K. Burnecki, G. Kukla D., and Taylor. *Pricing of Catastrophe Bond in Statistical Tools for Finance and Insurance*, (P. Cizek, W. Haerdle and R. Weron, eds.). Springer, Berlin, 2005.
- S. Campbell and F. Diebold. Weather forecasting for weather derivatives. *Journal of American Statistical Association*, 100(469):6–16, 2005.
- M. Cao and J. Wei. Weather derivatives valuation and market price of weather risk. *The Journal of Future Markets*, 24(11):1065–1089, 2004.
- A. Cartea and M. G. Figueroa. Pricing in electricity markets: a mean reverting jump diffusion model with seasonality. *Appl. Math. Finance*, 12(4):313–335, 2005.
- A. Cartea and T. Williams. Uk gas markets: The market price of risk and applications to multiple interruptible supply contracts. *To appear in Energy Economics*, 2007.
- Y. Chen, W.K. Haerdle, and U. Pigorsch. Localizing realized volatility modelling. *Re-submitted to Journal of the American Statistical Association*, 2010.
- P. Cizek, W. K. Haerdle, and V. Spokoiny. Adaptive pointwise estimation in time-inhomogeneous conditional heteroscedasticity models. *The Econometrics Journal*, 12: 248–271, 2009.
- R. Clarke, J. Faust, and C. McGhee. The catastrophe bond market at year-end 2004: The growing appetite for catastrophic risk. *Guy Carpenter and Company, Inc.*, 2005.
- R. Clarke, J. Faust, and C. McGhee. The catastrophe bond market at year-end 2005: Ripple effects from record storms. *Guy Carpenter and Company, Inc.*, 2006.
- R. Clarke, J. Faust, and C. McGhee. The catastrophe bond market at year-end 2006: Ripple into waves. *Guy Carpenter and Company, Inc.*, 2007.
- CME. An introduction to cme weather products. *CME Alternative Investment Products*, 2005.

## Bibliography

- D. C. Croson and H. C. Kunreuther. Customizing indemnity contracts and indexed cat bonds for natural hazard risks. *Journal of Risk Finance*, 1(3):24–41, 2000.
- J.D. Cummins. Cat bonds and other risk-linked securities: State of the market and recent developments. *Risk Management and Insurance Review*, 1(11):243–47, 2008.
- J.D. Cummins and P. Trainar. Securitization, insurance, and reinsurance. *Journal of Risk and Insurance*, 76(3):463–492, 2009.
- J.D. Cummins and M.A. Weiss. Convergence of insurance and financial markets: Hybrid and securitized risk transfer solutions. *Journal of Risk and Insurance*, 76(3):493–545, 2009.
- J.D. Cummins, D. Lalonde, and R.D. Phillips. The basis risk of catastrophic-loss index securities. *Journal of Financial Economics*, 71:77–111, 2004.
- R.B. D’Agostino and M.A. Stephens. *Goodness-of-Fit Techniques*. Marcel Dekker, New York, 1986.
- M.H.A. Davis. Pricing weather derivatives by marginal value. *Quantitative Finance*, 1: 305–308, 2001.
- Servicio Sismológico Nacional Instituto de Geosifísica UNAM. Earthquakes data base 1900-2003. *UNAM, Mexico*, 2006.
- Secretaría de Hacienda y Crédito Público México. Acuerdo que establece las reglas de operación del fondo de desastres naturales fonden. *SHCP, Mexico*, 2001.
- Secretaría de Hacienda y Crédito Público México. Administración de riesgos catastróficos del fonden. *SHCP*, 2004.
- M. Degen and P. Embrechts. EVT-based estimation of risk capital and convergence of high quantiles. *Adv. Appl. Prob.*, 40:696–715, 2008.
- F. X. Diebold and A. Inoue. Long memory and regime switching. *Journal of Econometrics*, 105:131–159, 2001.
- N. A. Doherty. *Innovation in Corporate Risk Management: The Case of Catastrophe Risk in Handbook of Insurance*, G.Dionne, ed. MA: Kluwer Academic, Boston, 2000.
- C. Dosi and M. Moretto. Global warming and financial umbrellas. *The Journal of Risk Finance*, 2003.
- W. Dubinsky and D. Laster. Insurance link securities. Technical report, Swiss Re Capital Markets Corporation, 2003.
- J. Dupacová, J. Hurt., and J. Štěpán. *Stochastic Modelling in Economics and Finance*. Kluwer Academic Publishers, 2002.

## Bibliography

- K. Dutta and J. Perry. A tail of tails: An empirical analysis of loss distribution model for estimating operational risk capital. *Working paper, Federal Reserve Bank of Boston*, 6 (13), 2007.
- F. Esscher. On the probability function in the collective theory of risk. *Skandinavisk Aktuarietidskrift*, 15:175–195, 1932.
- E.F. Fama. Efficient markes: a review of theory and empirical work. *Journal of Finance*, 25(2):383–417, 1970.
- M. Fengler, W. K. Haerdle, and E. Mammen. A dynamic semiparametric factor model for implied volatility string dynamics. *Financial Econometrics*, 5(2):189–218, 2007.
- S. Finken and C. Laux. Catastrophe Bonds and Reinsurance: The Competitive Effect of Information-Insensitive Triggers. *Journal of Risk and Insurance*, 76(3):579–605, 2009.
- K.A. Froot. The market for catastrophe risk: A clinical examination. *Journal of Risk Financial Economics*, 60:529–571, 2001.
- H. U. Gerber and E. S. W Shiu. Option pricing by esscher transforms. *Trans. Soc. Actuaries*, 46:99–191, 1994.
- J. Grandell. *Aspects of Risk Theory*. Springer, New York, 1991.
- C.W.J. Granger and N. Hyung. Occasional structural breaks and long memory with an application to the s&p 500 absolute stock returns. *Journal of Empirical Finance*, 11: 399–421, 2004.
- P. Grossi and H. Kunreuther. *Catastrophe modelling : a new approach to managing risk*. Springer, Huebner International Series on Risk, Insurance and Economic Security, Vol. 25, New York, 2005.
- W.K. Haerdle and B. López-Cabrera. Calibrating of parametric cat bonds: A case study of mexican earthquakes. *Journal of Applied Social Sciences Studies, Schmollers Jahrbuch, Duncker & Humblot, Berlin*, 128:615–630, 2008.
- W.K. Haerdle and B. López-Cabrera. Calibrating cat bonds for mexican earthquakes. *Journal of Risk and Insurance*, 2010a.
- W.K. Haerdle and B. López-Cabrera. Statistical analysis of the implied market price of weather risk. *Invitation to resubmission in Applied Mathematical Finance*, 2010b.
- W.K. Haerdle, B. López-Cabrera, and W. Wang. Localizing temperature residuals. *Working paper, Humboldt Universitaet zu Berlin*, 2010.
- D. Heath. Bond pricing and the term structure of interest rates: a new methodology for contingent claim valuation. *Econometrica*, 60:77–105, 1992.
- R.V. Hogg and S.A. Klugman. *Loss distributions*. Wiley, New York, 1984.



## Bibliography

- D. C. Howell. *Treatment of missing data*. URL, 1998. URL <http://www.uvm.edu/~dhowell/StatPages/>.
- H. Hung-Hsi, S. Yung-Ming, and L. Pei-Syun. Hdd and cdd option pricing with market price of weather risk for taiwan. *The Journal of Future Markets*, 28(8):790–814, 2008.
- N. Ikeda and S. Watanabe. *Stochastic Differential Equations and Diffusion Processes*. North-Holland/Kodansha, 1981.
- IAIS International Association of Insurance Supervisors. Non-life insurance securitisation. Technical report, IAIS Issues papers and other reports, 2003.
- S. Jewson, A. Brix, and C. Ziehmman. *Weather Derivative valuation: The Meteorological, Statistical, Financial and Mathematical Foundations*. Cambridge University Press, 2005.
- I. Karatzas and S. Shreve. *Brownian Motion and Stochastic Calculus*. Springer Verlag, New York, 1991.
- I. Karatzas and S. Shreve. *Methods of Mathematical Finance*. Springer Verlag, New York, 2001.
- I. Karatzas, D.L. Ocone, and J. Li. An extension of clark’s formula. *Stochastic Stochastic Rep.*, 37(3):215–258, 1991.
- J.B. Kau and D.C. Keenan. An option-theoretic model of catastrophes applied to mortgage insurance. *Journal of Risk and Insurance*, 63(4):639–656, 1996.
- R. W. Klein and S. Wang. Catastrophe risk financing in the us and the eu: A comparative analysis of alternative regulatory approaches. *Journal of Risk and Insurance*, 76(3):607–637, 2009.
- S.A. Klugman, H.H. Panjer, and G.E. Willmot. *Loss Models: From Data to Decisions*, 3rd. ed. John Wiley and Sons, New York, 2008.
- J.-P. Lee and M.-T. Yu. Pricing default-risky cat bonds with moral hazard and basis risk. *Journal of Risk and Insurance*, 69(1):25–44, 2002.
- J.-P. Lee and M.-T. Yu. Valuation of catastrophe reinsurance with cat bonds. *Insurance: Mathematics and Economics*, 41(2):264–278, 2007.
- R.S Liptser and A.N. Shiryaev. *Statistics of Random Processes I. General Theory*. Springer-Verlag, 1991.
- P. Malliavin and A. Thalmaier. *Stochastic Calculus of variations in Mathematical Finance*. Springer-Verlag, Berlin, Heidelberg, 2006.
- C. McGhee. The catastrophe bond market at year-end 2003: Market update. *Guy Carpenter and Company, Inc.*, 2004.

## Bibliography

- D. Mercurio and V. Spokoiny. Statistical inference for time-inhomogeneous volatility models. *The Annals of Statistics*, 32(2):577–602, 2004.
- S. Mooney. The world catastrophe reinsurance market 2005. *Guy Carpenter and Company, Inc.*, 2005.
- M. Mraoua and D. Bari. Temperature stochastic modelling and weather derivatives pricing: empirical study with moroccan data. *Afrika Statistika*, 2(1):22–43, 2007.
- M. Musiela and M. Rutkowski. *Martingale Methods for Financial Modelling*. Springer-Verlag, Berlin, Heidelberg, 1997.
- Working Group on California Earthquake Probabilities. Probabilities of large earthquakes occurring in california on the san andreas fault. *U.S. Geol. Open-file report*, 398:1–62, 1988.
- E. Platen and J. West. A fair pricing approach to weather derivatives. *Asian-Pacific Financial Markets*, 11(1):23–53, 2005.
- PricewaterhouseCoopers. Results of the 2005 pwc survey, presentation to weather risk managment association by john stell, <http://www.wrma.org>. *PricewaterhouseCoopers LLP*, 2005.
- P. Protter. *Stochastic Integration and Differential Equations*. Springer-Verlag, 1990.
- T. Richards, M. Manfredo, and D. Sanders. Pricing weather derivatives. *American Journal of Agricultural Economics*, 86(4):1005–1017, 2004.
- P. Samuelson. Rational theory of warrant pricing. *Indust. Manag. Rev.*, 6:13–32, 1965.
- E.S. Schwartz. The stochastic behaviour of commodity prices: Implications for valuation and hedging. *Journal of Finance*, 52(3), 1997.
- Moody’s Investors Service. Default and recovery rates: 1920-2006. Technical report, Moody’s Special Report, New York, 2007.
- S.E. Shreve. *Stochastic Calculus for Finance II, Continuous-Time Models*. Springer-Verlag, 2004.
- Risk Management Solutions. Mexico earthquake. RMS, [www.rms.com/Catastrophe/Models/Mexico.asp](http://www.rms.com/Catastrophe/Models/Mexico.asp), 2006.
- V. Spokoiny. Multiscale local change point detection with applications to value at risk. *The Annals of Statistics*, 37(3):1405–1436, 2009.
- Standard and Poor’s. Rating performance 2006: Stability and transition. *S&P, New York*, 2007.
- SwissRe. Capital market innovation in the insurance industry. *Sigma*, 3, 2001.

## *Bibliography*

SwissRe. New opportunities for insurers and investors. *Sigma*, 7, 2006.

V.E. Vaugirard. Valuing catastrophe bonds by monte carlo simulations. *Applied Mathematical Finance*, 10:75–90, 2003.

# List of abbreviations

a.s.	almost sure
cdf	Cumulative distribution function
e.g.	Exempli gratia; for example
et al.	Among others
i.e.	id est.; that is
r.v.	Random variable
$e_x$	Theoretical mean excess
$e_{n(x)}$	Empirical mean excess
$\exp(\cdot)$	Exponential
$f_x$	Probability distribution function
$\log(\cdot)$	Logarithm
$l_x$	Limited expected value function
$l_{n(x)}$	Empirical Limited expected value function
$B_t$	Brownian motion
$F_x$	Cumulative distribution function
$F_{n(x)}$	Empirical distribution function
$L_t$	Lévy process
$t$	Trading date
$\theta_t$	Market price of risk
$\tau$	Threshold time event
$\tau_1$	Start of measurement period
$\tau_1$	End of measurement period
$(\Omega, \mathcal{F}, \mathcal{F}_t, P)$	Probability space
$\phi(x)$	The standard normal pdf
$\Phi(x)$	The standard normal cdf
$I_p$	$p \times p$ Identity matrix
ACF	Autocorrelation function
AR	Autoregressive
CAR	Continuous autoregressive
CAT bond	Catastrophic bond
CAT	Cumulative Average Temperature
CBOT	Chicago Board of Trade
CME	Chicago Mercantile Exchange
CDD	Cooling Degree Days
C24AT	Accumulated total of 24-hour average temperature
FFT	Fourier Transform
FBM	Fraccional Brownian Motion
GARCH	Generalized Autoregressive conditional heteroskedastic

## *Bibliography*

HDD	Heating Degree Days
HJM	Heath-Jarrow-Morton
ILB	Insurance Linked Bonds
II	Independent increment
K	Strike price
LIBOR	London Interbank Offer rate
MPR	Market Price of Risk
OU	Ornstein-Uhlenbeck (process)
OTC	Over the counter, bilateral market
PACF	Partial Autocorrelation Functon
RP	Risk Premium
SPV	Special Purpose Vehicle
T	Expiration time
USD	US Dollars

# Selbständigkeitserklärung

Ich bezeuge durch meine Unterschrift, dass meine Angaben über die bei der Abfassung benutzten Hilfsmittel, über die mir zuteil gewordene Hilfe sowie über frühere Begutachtungen meiner Dissertation in jeder Hinsicht der Wahrheit entsprechen.

Berlin, den 16.03.2010

Brenda López Cabrera